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HEAT CONDUCTION

With Engineering and Geological Applications

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HEAT CONDUCTION

With Engineering and Geological Applications

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HEAT CONDUCTION

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PREFACE

The present volume is the successor to and, in effect, a revision of the Ingersoll and Zobel text of some years ago. To quote from the earlier preface: ". . . the theory of heat conduction is of importance, not only intrinsically but also because its broad bearing and the generality of its methods of analysis make it one of the best introductions to more advanced mathematical physics.

"The aim of the authors has been twofold. They have attempted, in the first place, to develop the subject with special reference to the needs of the student who has neither time nor mathematical preparation to pursue the study at great length. To this end, fewer types of problems are handled than in the larger treatises, and less stress has been placed on purely mathematical derivations such as uniqueness, existence, and convergence theorems.

"The second aim has been to point out . . . the many applications of which the results are susceptible . . . . It is hoped that in this respect the subject matter may be of interest to the engineer, for the authors have attempted to select applications with special reference to their technical importance, and in furtherance of this idea have sought and received suggestions from engineers in many lines of work. While many of these applications have doubtless only a small practical bearing and serve chiefly to illustrate the theory, . . . the results in some cases . . . may be found worthy of note. The same may be said of the geological problems.

"While a number of solutions are here presented for the first time . . . no originality can be claimed for the underlying mathematical theory which dates back, of course, to the time of Fourier."

Since the above was written there has been a steady increase

in interest in the theory of heat conduction, largely along practical lines. The geologist and geographer are interested in a new tool which will help them in explaining many thermal phenomena and in establishing certain time scales. The engineer, whose use of the theory was formerly limited almost entirely to the steady state, has developed many useful tables and curves for the solution of more general cases and is interested in finding still other methods of attack. The physicist and mathematician have done their part in treating problems which have hitherto resisted solution.

The present volume carries out and extends the aims of the earlier one. Most of the old material has been retained, although revised, and almost an equal amount of new has been added. The geologist, geographer, and engineer will find many new applications discussed, while the mathematician, physicist, and chemist will welcome the addition of a little Bessel function and conjugate function theory, as well as the several extended tables in the appendixes. Some of these are new and have had to be specially evaluated. The number of references has also been greatly enlarged and three-quarters of them are of more recent date than the older volume. A special feature is the extended treatment, particularly as regards applications, of the theory of permanent sources. This is carried out for all three dimensions, but most of the applications center about the two-dimensional case, the most interesting of these being the theory of ground-pipe heat sources for the heat pump. Other features of the revision are a modernized nomenclature, many new problems and illustrations, and the segregation of descriptions of methods of measuring heat-conduction constants in a special chapter.

A feature of particular importance to those whose interests are largely on the practical side is the discussion in Chapter 11 of auxiliary graphical and other approximation methods by which many practical heat conduction problems may be solved with only the simplest mathematics. It is believed that many will appreciate this and in particular the discussion of procedures by which it is possible to handle simply, and with sufficient accuracy for practical purposes, many problems whose

solution would be almost impossible by classical methods. As regards the book as a whole, the only mathematical prerequisite necessary for reading it is a reasonable knowledge of calculus. Despite occasional appearances to the contrary, the mathematical theory is not difficult and falls into a pattern which is readily mastered. The authors have tried, in general, to reduce mathematical difficulties to a minimum, and in some cases have deliberately chosen the simpler of two alternate methods of solving a problem, even at a small sacrifice of accuracy.

The authors acknowledge again their indebtedness to the several standard treatises referred to in the preface to the earlier edition, and in particular to Carslaw's "Mathematical Theory of the Conduction of Heat in Solids"; also Carslaw and Jaeger's "Conduction of Heat in Solids." It is hard to single out for special credit any of the hundred-odd other books and papers to which they are indebted and which are listed at the end of this volume, but perhaps particular reference should be made to McAdams' "Heat Transmission" and to papers by Emmons, Newman, and Olson and Schultz.

The authors are glad to acknowledge assistance from many friends. These include: O. A. Hougen, D. W. Nelson, F. E. Volk, and M. O. Withey of the College of Engineering, University of Wisconsin; J. D. MacLean of the Forest Products Laboratory; J. H. Van Vleck of Harvard University, W. J. Mead of Massachusetts Institute of Technology, and A. C. Lane of Cambridge; C. E. Van Orstrand, formerly of the U.S. Geological Survey; H. W. Nelson of Oak Ridge, Tennessee; C. C. Furnas of the Curtiss-Wright Corp., B. Kelley of the Bell Aircraft Corp., and G. H. Zenner and L. D. Potts of the Linde Air Products Laboratory, in Buffalo; A. C. Crandall of the Indianapolis Light and Power Co.; M. S. Oldacre of the Utilities Research Commission in Chicago; and a large number of others who have given help and suggestions. The authors are particularly indebted to F. T. Adler of the Department of Physics of the University of Wisconsin and to H. W. March of the Department of Mathematics for much assistance; also to K. J. Arnold of the same department and to Mrs. M. H. Glissendorf and Miss R. C. Bernstein of the university computing service

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THE AUTHORS

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CHAPTER 1

INTRODUCTION

1.1. Symbols. The following table lists the principal symbols and abbreviations used in this book. They have been chosen in agreement, so far as practicable, with the recommendations of the American Standards Association and with general scientific practice.

TABLE 1.1.—NOMENCLATURE

A	Area, cm^2 or ft^2.
α	Thermal diffusivity, cgs or fph (Secs. 1.3, 1.5, Appendix A).
$B(x)$	$2(e^{-x} - e^{-4x} + e^{-9x} - \cdots)$ (Sec. 9.17, Appendix H).
$B_a(x)$	$\frac{6}{\pi^2}\left(e^{-x} + \frac{1}{4}e^{-4x} + \frac{1}{9}e^{-9x} + \cdots\right)$ (Sec. 9.18, Appendix H).
β, γ	Variables of integration; also constants.
λ	Variable of integration; also a constant; also wave length.
Btu	British thermal unit, 1 lb water 1°F (Sec. 1.5).
c	Specific heat (constant pressure), cal/(gm)(°C), or Btu/(lb)(°F); also a constant.
cal	Calorie, 1 gm water 1°C (Sec. 1.5).
cgs	Centimeter-gram-second system; used here only with centigrade temperature scale and calorie as unit of heat.
$C(x)$	$2\left(\frac{e^{-xz_1^2}}{z_1J_1(z_1)} + \frac{e^{-xz_2^2}}{z_2J_1(z_2)} + \frac{e^{-xz_3^2}}{z_3J_1(z_3)} + \cdots\right)$ (Sec. 9.38, Appendix J).
exp x	e^x.
fph	Foot-pound-hour system, used here only with Fahrenheit temperature scale and Btu as heat unit.
h	Coefficient of heat transfer between a surface and its surroundings, cal/(sec)(cm^2)(°C) or Btu/(hr)(ft^2)(°F); sometimes called "emissivity" or "exterior conductivity" (Sec. 2.5, Appendix A).
η	$\frac{1}{2\sqrt{\alpha t}}$.
$I(x)$	$\int_x^\infty \beta^{-1}e^{-\beta^2}\,d\beta$ (Sec. 9.8, Appendix F).
$J_n(x)$	Bessel function (Sec. 9.36).
k	Thermal conductivity, cgs or fph (Secs. 1.3, 1.5, Appendix A).
ln x	$\log_e x$.

TABLE 1.1.—NOMENCLATURE—(*Continued*)

$\Phi(x)$	Probability integral, $\frac{2}{\sqrt{\pi}}\int_0^x e^{-\beta^2}\,d\beta$ (Appendix D).
Q	Quantity of heat, cal or Btu (sometimes taken per unit length or unit area; see Q').
q	Rate of heat flow, cal/sec or Btu/hr (sometimes also used for rate of heat production).
Q'	Rate of heat production or withdrawal in permanent sources or sinks, cal/sec or Btu/hr for three-dimensional case; cal/sec per cm length or Btu/hr per ft length for two-dimensional case; cal/(sec)(cm^2) or Btu/(hr)(ft^2) for one-dimensional case (Secs. 8.2, 9.9).
ρ	Density, gm/cm^3, or lb/ft^3.
R	Thermal resistance $\frac{x}{kA}$ (Sec. 3.3).
S	Strength of instantaneous source, $\frac{Q}{c\rho}$ (Secs. 8.2, 9.9).
S'	Strength of permanent source, $\frac{Q'}{c\rho}$ (Secs. 8.2, 9.9).
$S(x)$	$\frac{4}{\pi}\left(e^{-\pi^2x} - \frac{1}{3}e^{-9\pi^2x} + \frac{1}{5}e^{-25\pi^2x} - \cdots\right)$ (Sec. 8.16, Appendix G).
t	Time, seconds or hours.
T*	Temperature, °C or °F.
w	Rate of flow of heat per unit area, $\frac{q}{A}$; cal/(sec)(cm^2) or Btu/(hr)(ft^2) (Sec. 1.3)

1.2. Historical. The mathematical theory of heat conduction in solids, the subject of principal concern in this book, is due principally to Jean Baptiste Joseph Fourier (1768–1830) and was set forth by him in his "Théorie analytique de la chaleur."[42]† While Lambert, Biot, and others had developed some more or less correct ideas on the subject, it was Fourier who first brought order out of the confusion in which the experimental physicists had left the subject. While Fourier treated a large number of cases, his work was extended and applied to more complicated problems by his contemporaries Laplace and Poisson, and later by a number of others, including Lamé, Sir W. Thomson[146,147] (Lord Kelvin), and Riemann.[160] To the

* The use of θ for temperature, as in the former edition of this book, has been discontinued here, partly because many modern writers attach the significance of time to it and partly because of the increasing adoption of T. It is suggested that, to avoid confusion, this be always pronounced "captee."

† Superscript figures throughout the text denote references in Appendix M.

last mentioned writer all students of the subject should feel indebted for the very readable form in which he has put much of Fourier's work. The most authoritative recent work on the subject is that of Carslaw and Jaeger.[27a]

1.3. Definitions. When different parts of a solid body are at different temperatures, heat flows from the hotter to the colder portions by a process of electronic and atomic energy transfer known as "conduction." The rate at which heat will be transferred has been found by experiment to depend on a number of conditions that we shall now consider.

To help visualize these ideas imagine in a body two parallel planes or laminae of area A and distance x apart, over each of which the temperature is constant, being T_1 in one case and T_2 in the other. Heat will then flow from the hotter of these isothermal surfaces to the colder, and the quantity Q that will be conducted in time t will be given by

$$Q = k \frac{T_1 - T_2}{x} At \tag{a}$$

or

$$q = \frac{dQ}{dt} = k \frac{T_1 - T_2}{x} A \tag{b}$$

where k is a constant for any given material known as the *thermal conductivity* of the substance. It is then numerically equal to the quantity of heat that flows in unit time through unit area of a plate of unit thickness having unit temperature difference between its faces.

The limiting value of $(T_2 - T_1)/x$ or $\partial T/\partial x$ is known as the *temperature gradient* at any point. If due attention is paid to sign, we see that if $\partial T/\partial x$ is taken in the direction of heat flow it is intrinsically negative. Hence, if we wish to have a positive value for the rate at which heat is transferred across an isothermal surface in a positive direction, we write

$$q = -kA \frac{\partial T}{\partial x} \tag{c}$$

or

$$w = -k \frac{\partial T}{\partial x} \tag{d}$$

where w $(= q/A)$ is called the "flux" of heat across the surface

at that point. If instead of an isothermal surface we consider another, making an angle ϕ with it, we can see that both the flux across the surface and the temperature gradient across the normal to such surface will be diminished, the factor being cos ϕ, so that we may write in general for the flux across any surface

$$w = -k \frac{\partial T}{\partial n} \tag{e}$$

where the derivative is taken along the outward drawn normal, *i.e.*, in the direction of decreasing temperature. This shows that the direction of (maximum) heat flow is normal to the isotherms.

While the rate at which heat is transferred in a body, *e.g.*, along a thermally insulated rod, is dependent only on the conductivity and other factors noted, the rise in temperature that this heat will produce will vary with the specific heat c and the density ρ of the body. We must then introduce another constant α whose significance will be considered later, determined by the relation

$$\alpha = \frac{k}{c\rho} \tag{f}$$

The constant α has been termed by Kelvin the *thermal diffusivity* of the substance, and by Maxwell its *thermometric conductivity*.

Equations (a) and (e) express what is sometimes referred to as the fundamental hypothesis of heat conduction. Its justification or proof rests on the agreement of calculations made on this hypothesis, with the results of experiment, not only for the very simple but for the more complicated cases as well.

1.4. Fields of Application. From (1.3a) we may infer in what field the results of our study will find application. We may conclude first that our derivations will hold good for any body in which heat transfer takes place according to this law, if k is the same for all parts and all directions in the body. This includes all homogeneous isotropic solids and also liquids and gases in cases where convection and radiation are negligible. The equation also shows that, since only differences of temperature are involved, the actual temperature of the system is

immaterial. We shall have cause to remember this statement frequently; for, while many cases are derived on the supposition that the temperature at the boundary is zero, the results are made applicable to cases in which this is any other constant temperature by a simple shift of the temperature scale.

But the results of the study of heat conduction are not limited in their application to heat alone, for parts of the theory find application in certain gravitational problems, in static and current electricity, and in elasticity, while the methods developed are of very general application in mathematical physics. As an example of such relationship to other fields it may be pointed out that, if T in (1.3a) is interpreted as electric potential and k as electric conductivity, we have the law of the flow of electricity and all our derivations may be interpreted accordingly.

Another field of application is in drying of porous solids, *e.g.*, wood. It is found that for certain stages of drying the moisture flow is fairly well represented* by the heat-conduction equation. In this case Q represents the amount of water (or other liquid) transferred by diffusion, T is the moisture content in unit volume of the (dry) solid, k is the rate of moisture flow per unit area for unit concentration gradient. The quantity $c\rho$, which normally represents the amount of heat required to raise the temperature of unit volume of the substance by one degree, is here the amount of water required to raise the moisture content of unit volume by unit amount. This is obviously unity, so k and α are the same in this case; k is here called the "diffusion constant." The passage of liquid through a porous solid, as in drying, is a more complicated process than heat flow, and the application of conduction theory has definite limitations, as pointed out by Hougen, McCauley, and Marshall.[58] It may be added that in all probability the diffusion of gas in a metal is subject to the same general theory as water diffusion in porous materials.

Lastly, we may mention the work of Biot[15] on settlement and consolidation of soils. This indicates that the conduction

* Bateman, Hohf and Stamm,[8] Ceaglske and Hougen,[29] Gilliland and Sherwood,[45] Lewis,[85] McCready and McCabe,[91] Newman,[101] Sherwood,[127,128] and Tuttle.[150]

equation may play an important part in the theory of these phenomena.

1.5. Units; Dimensions. Two consistent systems of conductivity units are in common use, having as units of length, mass, time, and temperature, respectively, the centimeter, gram, second, and centigrade degree, on the one hand and the foot, pound, hour, and Fahrenheit degree on the other. The former unit will be referred to as cgs and the latter as fph as regards system. This gives as the unit of heat in the first case the (small) calorie, or heat required to raise the temperature of 1 gm of water 1°C, frequently specified at 15°C; and in the second the Btu, or heat required to raise 1 lb of water 1°F, sometimes specified at 39.1°F* and sometimes at 60°F. The cgs thermal-conductivity unit is the calorie per second, per square centimeter of area, for a temperature gradient of 1°C per centimeter, which shortens to cal/(sec)(cm)(°C), while the fph conductivity unit is the Btu/(hr)(ft)(°F). Similarly, the units of diffusivity come out cm^2/sec and ft^2/hr. The unit in frequent use in some branches of engineering having areas in square feet but temperature gradients expressed in degrees per *inch* will not be used here because of difficulties attendant on the use of two different units of length.

In converting thermal constants from one system to another and in solving many problems Table 1.2 will be found useful.

Conversion factors other than those listed above may be readily derived from a consideration of the dimensions of the units. From (1.3*a*)

$$k = \frac{Q}{T_1 - T_2} \frac{x}{At} \qquad (a)$$

Since—putting the matter as simply as possible—the unit of heat is that necessary to raise unit mass of water one degree, its dimensions are mass and temperature; thus, the dimensions of $Q/(T_1 - T_2)$ are simply M. Hence, K the unit of conductivity is the unit of mass M divided by the units of length L

* The matter of whether heat units are specified for the temperature of maximum density of water or for a slightly higher temperature may result in discrepancies of the order of half a percent, but this is of little practical importance since this is below the usual limit of error in thermal conductivity work.

TABLE 1.2.—CONVERSION FACTORS AND OTHER CONSTANTS

$$
\begin{aligned}
1 \text{ m} &= 39.370 \text{ in.} = 3.2808 \text{ ft} = 1.0936 \text{ yd} \\
1 \text{ in.} &= 2.540 \text{ cm} \\
1 \text{ ft} &= 30.48 \text{ cm} \\
1 \text{ m}^2 &= 10.764 \text{ ft}^2 = 1.196 \text{ yd}^2 \\
1 \text{ in.}^2 &= 6.452 \text{ cm}^2 \\
1 \text{ ft}^2 &= 929.0 \text{ cm}^2 \\
1 \text{ m}^3 &= 61{,}023 \text{ in.}^3 = 35.314 \text{ ft}^3 = 1.308 \text{ yd}^3 \\
1 \text{ in.}^3 &= 16.387 \text{ cm}^3 \\
1 \text{ ft}^3 &= 28{,}317 \text{ cm}^3 \\
1 \text{ kg} &= 2.2046 \text{ lb} \\
1 \text{ lb} &= 453.6 \text{ gm} \\
1 \text{ gm/cm}^3 &= 62.4 \text{ lb/ft}^3 \\
1 \text{ Btu} &= 252 \text{ cal} = 1055 \text{ joules} = 777.5 \text{ ft-lb} \\
1 \text{ watt} &= 0.2389 \text{ cal/sec} \\
1 \text{ kw} &= 56.88 \text{ Btu/min} = 3413 \text{ Btu/hr} \\
1 \text{ cal} &= 4.185 \text{ joules} \\
1 \text{ cal/cm}^2 &= 3.687 \text{ Btu/ft}^2 \\
1 \text{ cal/sec} &= 14.29 \text{ Btu/hr} \\
1 \text{ watt/ft}^2 &= 3.413 \text{ Btu/(ft}^2\text{)(hr)} \\
1 \text{ cal/(cm}^2\text{)(sec)} &= 318{,}500 \text{ Btu/(ft}^2\text{)(day)} \\
1 \text{ Btu/hr} &= 0.293 \text{ watts} = 0.000393 \text{ hp} \\
1 \text{ yr} &= 3.156 \times 10^7 \text{ sec} = 8{,}766 \text{ hr} \\
k \text{ in fph} &= 241.9\, k \text{ in cgs} \\
k \text{ in cgs} &= 0.00413\, k \text{ in fph} \\
\alpha \text{ in fph} &= 3.875\, \alpha \text{ in cgs} \\
\alpha \text{ in cgs} &= 0.2581\, \alpha \text{ in fph} \\
\text{Temp C} &= \tfrac{5}{9}(\text{temp F} - 32) \\
e &= 2.7183 = 1/0.36788 \\
\pi &= 3.1416 = 1/0.31831 \\
\pi^2 &= 9.8696 = 1/0.10132 \\
\sqrt{\pi} &= 1.7725 = 1/0.56419 \\
g \text{ (45° lat)} &= 980.6 \text{ cm/sec}^2 = 32.17 \text{ ft/sec}^2
\end{aligned}
$$

and time θ. If we have another system in which the units are M', L', and θ', the number k' that represents the conductivity in this system is related to the number k that represents the conductivity in the first system, through the equation

$$k \frac{M}{L\theta} = k' \frac{M'}{L'\theta'} \tag{b}$$

or

$$k' = k \frac{M}{M'} \frac{L'}{L} \frac{\theta'}{\theta} \tag{c}$$

Similarly, it is easily shown that for diffusivity

$$\alpha' = \alpha \frac{L^2\theta'}{L'^2\theta} \tag{d}$$

1.6. Values of the Constants. In Appendix A is given a table of the conductivity coefficients, or "constants," as they are called—even if they show considerable variation with temperature and other factors—for a considerable number of substances, in both cgs and fph units. Thermal conductivities of different solids at ordinary temperatures range in value some 20,000 fold. Of ordinary materials silver (k = 0.999 cgs or 242 fph) is the best conductor,* with copper only slightly inferior and iron hardly more than one-tenth as good. Turning to the poor conductors or insulators, we have materials ranging from certain rocks with conductivities around 0.005 cgs vs. 1.2 fph, down to silica aerogel, whose conductivity of 0.00005 cgs vs. 0.012 fph is actually a little less than that measured for still air. A considerable number of building insulators have values in the neighborhood of 0.0001 cgs vs. 0.024 fph. Loosely packed cotton and wool are also in this category. Because of density and specific-heat considerations the diffusivities follow the order of conductivities only in a general way, in some cases being strikingly out of line. The range is smaller, running from 1.7 cgs vs. 6.6 fph for silver, down to about 0.0008 cgs vs. 0.003 fph for soft rubber.

Of the factors affecting conductivity one of the most important for porous, easily compressible materials such as cotton, wool, and many building insulators is the degree of compression or bulk density. The ideal of such insulators is to break down the air spaces to a point where convection is negligible, in other words to approach the conductivity of air itself as closely as possible—and with a minimum of heat transmitted by radiation. Many building insulators come within a factor of two or three of this, for suitable bulk densities, and silica aerogel is actually below air as a conductor as already indicated. The question of density is one of the reasons why wool is, in practice, a better

* The remarkable substance liquid helium II has an apparent conductivity many thousands of times greater than silver; see Powell.[113, p. 179]

insulating material than cotton for clothing, bedding, etc. The difference between the two when new is small, but in use cotton tends to compact while wool keeps its porosity even in the presence of moisture.

Most metals show a small and nearly linear decrease of conductivity with increase of temperature, of the order of a few per cent per 100°C, but a few (*e.g.*, aluminum and brass) show the reverse effect as do also many alloys. The conductivity of nonmetallic substances increases in general with temperature (there are, however, many exceptions such as most rocks).[16] The diffusivity for such substances, however, usually shows a smaller change, as the specific heat in most cases also increases with temperature while the density change is small. When possible, the change of thermal constants with temperature should be taken into account in calculations, and this may be done approximately by using the conductivity and diffusivity for the average temperature involved. When k is linear with temperature, as is often the case, its arithmetic mean value for the two extreme temperatures can usually be used. If k is not linear, we can use a mean value k_m defined by

$$k_m(T_2 - T_1) = \int_{T_1}^{T_2} k\, dT \qquad (a)$$

In the more complicated cases of heat flow involving other than the steady state, it may be difficult to take into account temperature changes of thermal constants in a satisfactory manner.*

The modern theory of heat conduction in solids† involves the transmission of thermal agitation energy from hot to cold regions by means of the motion of free electrons and also through vibrations of the crystal lattice structure at whose lattice points the atoms (or ions) are located. The first part, or electronic contribution, is the most important for metals, and the second part for nonmetallic solids.

Because of the predominantly electronic nature of metallic conduction it might be expected that there would be a relation between the thermal and electrical conductivities of metals, and this fact is expressed in the law of Wiedemann and Franz

* See Sec. 11.20 for the solution of a special problem involving such changes.

† See, *e.g.*, Austin,[2] Hume-Rothery,[59] and Seitz.[125]

that states that one is proportional to the other. While this holds in a general way where different metals are under consideration, it does not express the facts when a single metal at several different temperatures is concerned; for the electrical conductivity decreases with rise of temperature, while the thermal conductivity is more nearly constant. Lorenz[86] took account of this fact and expressed it in the law that the ratio of thermal divided by electrical conductivity increases for any given metal proportionately to the absolute temperature. It holds only for pure metals with any degree of approximation and only for very moderate temperature ranges. Griffiths,[50] however, finds that this law holds also for certain aluminum and bronze alloys.

CHAPTER 2

THE FOURIER CONDUCTION EQUATION

2.1. Differential Equations. In any mathematical study of heat conduction use must continually be made of differential equations, both ordinary and partial. These occur, however, only in a few special forms whose solutions can be explained as they appear, so only a brief general discussion of the subject is necessary here.

Differential equations are those involving differentials or differential coefficients and are classified as *ordinary* or *partial*, according as the differential coefficients have reference to one, or to more than one, independent variable. A solution of such an equation is a function of the independent variables that satisfies the equation for all values of these variables. For example,

$$y = \sin x + c \tag{a}$$

is a solution of the simple differential equation

$$dy = \cos x \, dx \tag{b}$$

The general solution, as its name implies, is the most general function of this sort that satisfies the differential equation and will always contain arbitrary, *i.e.*, undetermined, constants or functions. A particular solution may be obtained by substituting particular values of the constants or functions in the general solution. But while this is theoretically the method of obtaining the particular solution, we shall find in practice that in many cases where it would be almost impossible to obtain the general solution of the differential equation, we are still able to arrive at the desired result by combining particular solutions that can be obtained directly by various simple expedients.

2.2. A differential equation is *linear* when it is of the first degree with respect to the dependent variable and its deriva-

tives. It is also *homogeneous* if, in addition, there is no term that does not involve this variable or one of its derivatives. Practically all the differential equations we shall have occasion to use are both linear and homogeneous, as are indeed a large share of those occurring in all work in mathematical physics. As examples we may mention the following partial differential equations that are both linear and homogeneous:

Laplace's equation, of constant use in the theory of potential,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \qquad (a)$$

also the equation of the vibrating cord,

$$\frac{\partial^2 y}{\partial t^2} = b^2 \frac{\partial^2 y}{\partial x^2} \qquad (b)$$

and the Fourier conduction equation,

$$\frac{\partial T}{\partial t} = \alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) \qquad (c)$$

2.3. The Fourier Equation. We shall now derive this last equation. Choose three mutually rectangular axes of reference OX, OY, and OZ (Fig. 2.1) in any isotropic body and consider a small rectangular parallelepiped of edges Δx, Δy, and Δz parallel, respectively, to these three axes. Let T denote the temperature at the center of this element of volume; then, since the temperature will in general be variable throughout the body, we may express its value on any face of the parallelepiped—this being so small that the temperature is effectively uniform over any one face—as being greater or less than this mean temperature T by a small amount. The magnitude of this small amount for the case of the $\Delta y \Delta z$ faces we may readily show to be

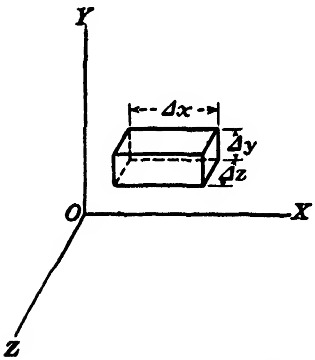

FIG. 2.1. Elementary parallelepiped in medium through which heat is flowing.

$$\frac{1}{2} \frac{\partial T}{\partial x} \Delta x \qquad (a)$$

since the temperature gradient $\partial T/\partial x$ measures the change of temperature per unit length along OX, and the distance of $\Delta y \Delta z$ from the center is evidently $\frac{1}{2}\Delta x$. Then the temperature of the left- and right-hand faces may be written

$$T_L = T - \frac{1}{2}\frac{\partial T}{\partial x}\Delta x, \qquad T_R = T + \frac{1}{2}\frac{\partial T}{\partial x}\Delta x \tag{b}$$

Using (1.3*c*), $q = -kA\partial T/\partial x$, we see that the flow of heat per second in the positive x direction through the left-hand face $\Delta y \Delta z$ is

$$q_L = -k\Delta y \Delta z \frac{\partial}{\partial x}\left(T - \frac{1}{2}\frac{\partial T}{\partial x}\Delta x\right) \tag{c}$$

and through the right-hand face in the same direction

$$q_R = -k\Delta y \Delta z \frac{\partial}{\partial x}\left(T + \frac{1}{2}\frac{\partial T}{\partial x}\Delta x\right) \tag{d}$$

the negative sign being used, since a positive flow of heat evidently requires a negative temperature gradient. The difference between these two quantities is evidently the gain in heat of the element due to the x component of flow alone; then, since similar expressions hold for the other two pairs of faces, the sum of the differences of these three pairs of expressions, or

$$k\frac{\partial^2 T}{\partial x^2}\Delta x \Delta y \Delta z + k\frac{\partial^2 T}{\partial y^2}\Delta x \Delta y \Delta z + k\frac{\partial^2 T}{\partial z^2}\Delta x \Delta y \Delta z \tag{e}$$

represents the difference between the total inflow and total outflow of heat, or the amount by which the heat of the element is being increased per second. If the specific heat of the material of the body is c and its density ρ, this sum must equal

$$c\rho\Delta x \Delta y \Delta z \frac{\partial T}{\partial t} \tag{f}$$

Hence, we may write

$$k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) = c\rho\frac{\partial T}{\partial t} \tag{g}$$

or, since $\alpha \equiv k/c\rho$,

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) \tag{h}$$

which is usually written

$$\frac{\partial T}{\partial t} = \alpha\nabla^2 T^* \tag{i}$$

This is known as *Fourier's equation.* It expresses the conditions that govern the flow of heat in a body, and the solution of any particular problem in heat conduction must first of all satisfy this equation, either as it stands or in a modified form.

In the general case, where the thermal conductivity varies from point to point, the corresponding equation is†

$$\frac{\partial T}{\partial t} = \frac{1}{c\rho}\left[\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)\right] \tag{j}$$

Its solution would be more difficult than that of the previous one.

2.4. If a linear and homogeneous equation such as the Fourier equation is written so that all the terms are on the left side, the right-hand member being consequently reduced to zero, a very useful proposition can be deduced at once as follows: Any value of the dependent variable that satisfies the equation must reduce the left-hand member to zero. Thus, if such particular solution is multiplied by a constant, it will still reduce this member to zero, as this is merely equivalent to multiplying each term by the constant. In the same way it can be seen that the sum of any number of particular solutions will still be a solution. We may then state as a general proposition that, *in the case of the linear, homogeneous differential equation (ordinary or partial), any combination formed by adding particular solutions, with or without multiplication by arbitrary constants, is still a solution.* We shall have frequent occasion to make application of this law.

2.5. Boundary Conditions. The solution of practically all heat-conduction problems involves the determination of the temperature T as a function of the time and space coordinates. Such value of T is assumed to be a finite and continuous function of x,y,z and t and must satisfy not only the general differential equation, which in one modification or another is common to all

* ∇ is frequently called "nabla."

† See Bateman,[9, p. 120] Carslaw and Jaeger.[27a, p. 9]

heat-conduction problems, but also certain equations of condition that are characteristic of each particular problem. Such are

Initial Conditions. These express the temperature throughout the body at the instant that is chosen as the origin of the time coordinate, as a function of the space coordinates, *i.e.*,

$$T = f(x,y,z) \qquad \text{when } t = 0 \tag{a}$$

Boundary or Surface Conditions. These are of several sorts according as they express

1. The temperature on the boundary surface as a function of time, position, or both, *i.e.*,

$$T = \psi(x,y,z,t) \tag{b}$$

2. That at the surface of separation of two media there is continuity of flow of heat, expressed by the relation

$$k_1 \frac{\partial T_1}{\partial n} = k_2 \frac{\partial T_2}{\partial n}\text{*} \tag{c}$$

3. That the boundary surface is impervious to heat, expressed by

$$\left(\frac{\partial T}{\partial n}\right)_s = 0 \tag{d}$$

4. That radiation and convection losses take place at the surface, in which case we have, for surroundings at zero,

$$-k\left(\frac{\partial T}{\partial n}\right)_s = hT\dagger \tag{e}$$

In (*e*) h is the coefficient of heat transfer between the surface and surroundings (sometimes referred to as the emissivity or

* See (1.3*e*).

† This assumes Newton's law of cooling, which states that the rate of loss of heat is proportional to the temperature above the surroundings, for small temperature differences. That this is not inconsistent with Stefan's law of radiation is shown by the following simple reasoning: Stefan's law states that radiation $q_r = C(K^4 - K_0^4)$, where K and K_0 are the absolute temperatures of the radiating body and of the surrounding walls, respectively. For small values of $K - K_0$ we have $K^4 - K_0^4 = \Delta(K^4) = 4K_0^3\Delta K$, or $q_r = 4CK_0^3\Delta K$, which agrees with (*e*) if we remember that ΔK is here equivalent to T.

as the exterior or surface conductivity*), *i.e.*, the rate of loss of heat by radiation and convection per unit area of surface per degree above the temperature of the surroundings. h is a constant only for relatively small temperature differences.

There are also other possible boundary conditions, which we shall have frequent occasion to use and shall treat more at length when they occur. Following a common practice, we shall hereafter refer to both initial and surface conditions as simply "boundary conditions."

2.6. Uniqueness Theorem. Our task in general, then, in solving any given heat-conduction problem is to attempt, by building up a combination of particular solutions of the general conduction equation, to secure one that will satisfy the given boundary conditions. It is easy to see that such a result is *one* solution of our problem and it may be shown that it is also the *only* solution. The reader is referred to the larger treatises (*e.g.*, Carslaw[27]) for a rigorous proof of this uniqueness theorem, but the following simple physical discussion is satisfactory for our purposes:

Consider a solid body with the Fourier equation (2.3*i*) holding everywhere inside, with the initial condition

$$T = f(x,y,z) \qquad \text{for } t = 0 \tag{a}$$

and the boundary condition

$$T = \psi(x,y,z,t) \qquad \text{at the surface} \tag{b}$$

Assume that there are two solutions T_1 and T_2 of these equations, and let $\theta \equiv T_1 - T_2$. Then θ satisfies

$$\frac{\partial \theta}{\partial t} = \alpha \nabla^2 \theta \tag{c}$$

and, since T_1 and T_2 are obviously equal under the conditions (*a*) and again of (*b*),

$$\theta = 0 \text{ for } t = 0 \qquad \text{in the solid} \tag{d}$$

and

$$\theta = 0 \qquad \text{at the surface} \tag{e}$$

We shall now visualize these last three equations as temperature equations applying to some body. The two boundary

* See Carslaw and Jaeger.[27a, p. 13]

conditions mean that the temperature is initially everywhere zero inside the body and that it is at all times zero at the surface. Now it is physically impossible for an isolated body whose initial temperature is everywhere zero and whose surface is kept at zero ever to be other than zero at any point—radiation and self-generation of heat, of course, excluded. In other words, $\theta = 0$ throughout the volume and for any time, which means that the two assumed solutions T_1 and T_2 are the same.

CHAPTER 3

STEADY STATE—ONE DIMENSION

3.1. A body in which heat is flowing is said to have reached a *steady state* when the temperatures of its different parts do not change with time. Such a state occurs in practice only after the heat has been flowing for a long while. Each part of the body then gives up on one side as much heat as it receives on the other, and the temperature is therefore independent of the time t, although it varies from point to point in the body, being a function of the coordinates x, y, and z. For the steady state, then, Fourier's equation (2.3h) becomes

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0 \qquad (a)$$

We shall investigate a few applications of this equation for the case of flow in the x direction only.

3.2. One-dimensional Flow of Heat. This includes the common cases of flow of heat through a thin plate or along a rod, the two faces of the plate, or ends of the rod, being at constant temperatures T_1 and T_2, and in the latter case the surface of the rod being protected so that heat can enter or leave only at the ends. It also includes the case of the steady flow of heat in any body such that the isothermal surfaces, or surfaces of equal temperature, are parallel planes.

For these cases the general equation of conduction reduces to

$$\frac{d^2 T}{dx^2} = 0 \qquad (a)$$

the ordinary derivative being written instead of the partial, since in the case of only a single independent variable a partial derivative would have no particular significance. This integrates into

$$T = Bx + C \qquad (b)$$

The constants B and C are determined from the boundary conditions for this case, which are that the temperature is T_1 at the face of the plate (or end of the bar) whose distance from the yz plane may be called l, and T_2 for the face at distance m; or, as these conditions may be simply expressed,

$$T = T_1 \text{ at } x = l; \qquad T = T_2 \text{ at } x = m \tag{c}$$

Therefore, $T_1 = Bl + C$ and $T_2 = Bm + C$. Evaluating B and C, we get as the temperature at any point in a plate distant x from the yz plane

$$T = \frac{mT_1 - lT_2}{m - l} - \frac{(T_1 - T_2)x}{m - l} \tag{d}$$

This, with the aid of (1.3d), gives

$$w = \frac{k(T_1 - T_2)}{m - l} = k\frac{T_1 - T_2}{u} \tag{e}$$

where u is the thickness of the plate or length of the rod. This, of course, also follows directly from (1.3b).

3.3. Thermal Resistance. The close relationship between thermal and electrical equations suggests at once that the concept of thermal resistance may be useful. Thus, (1.3b) may be written (overlooking the minus sign)

$$q = k\frac{A\Delta T}{x} = \frac{\Delta T}{x/kA} = \frac{\Delta T}{R} \tag{a}$$

where

$$R \equiv \frac{x}{kA} \tag{b}$$

is called the *thermal resistance.** It is particularly useful in the case of steady heat flow through several layers of different thickness and conductivity in series (Fig. 3.1a). Here (again overlooking sign)

$$q = \frac{T_2 - T_1}{R_a} = \frac{T_3 - T_2}{R_b} = \frac{T_4 - T_3}{R_c} \tag{c}$$

or $\quad T_2 - T_1 = qR_a; \quad T_3 - T_2 = qR_b; \quad T_4 - T_3 = qR_c \quad (d)$

* Some engineers use the concept of thermal *resistivity*, the reciprocal of conductivity. It is numerically equal to the resistance of a unit cube. In this case, however, the heat rate is usually measured in watts instead of cal/sec.

from which we get by addition

$$T_4 - T_1 = q(R_a + R_b + R_c) = qR \tag{e}$$

or
$$q = \frac{T_4 - T_1}{R} = \frac{T_4 - T_1}{(x_a/k_aA_a) + (x_b/k_bA_b) + (x_c/k_cA_c)} \tag{f}$$

This takes the general form

$$q = \frac{T_n - T_m}{\displaystyle\int_m^n \frac{dx}{kA}} \tag{g}$$

With the aid of (*f*) and (*d*) the temperatures T_2 and T_3 as in Fig. 3.1*a* may be readily computed. For a plane wall the areas

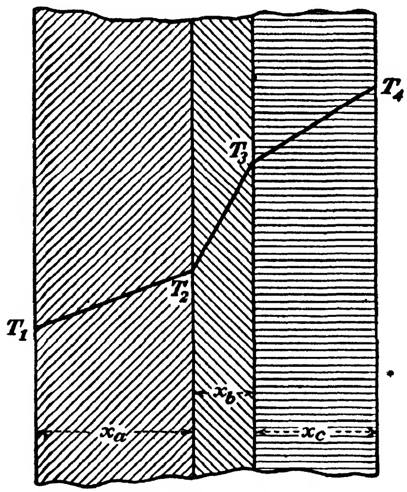

FIG. 3.1*a*. Temperature distribution in a composite wall; thermal resistances in series. (The heat flow is obviously to the left here.)

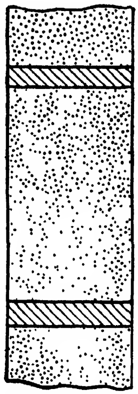

FIG. 3.1*b*. Wall with "through metal"; thermal resistances in parallel.

A_a, A_b, etc., are equal, but in many cases this will not be true, *e.g.*, when these considerations are applied to spherical or cylindrical flow (see Sec. 4.7).

The resistance concept is also useful when conductors, instead of being in series as above, are in parallel, as in an insulated wall with "through metal," *e.g.*, bolts extending from one side to the other (Fig. 3.1*b*). In this case

$$q_1 = \frac{T_2 - T_1}{R_a}, \qquad q_2 = \frac{T_2 - T_1}{R_b} \tag{h}$$

or

$$q = q_1 + q_2 = \frac{T_2 - T_1}{R} \tag{i}$$

where

$$\frac{1}{R} = \frac{1}{R_a} + \frac{1}{R_b} \tag{j}$$

Thus, an insulated wall of thickness x and conductivity of insulation 0.03 fph, with 0.2 per cent of its area consisting of iron bolts of conductivity 35 fph, may be readily shown from (i) to have no more insulation value than a wall without such bolts and of thickness only $0.3x$; *i.e.*, the heat loss is more than tripled by the presence of the bolts. Paschkis and Heisler find that the heat loss may be even more than that calculated in this way.

3.4. Edges and Corners.* If, in calculating the heat loss or gain from a furnace or refrigerator, we use A as the inside area, it is evident that the results will be much too low because of the loss through the edges and corners. The situation is no better if we use the outside area or even the arithmetic mean area, for in this case the calculated values are too high. If the lengths y of the inside edges are each greater than about one-fifth† the thickness x of the walls, the work of Langmuir, Adams, and Meikle[81] gives this equation for the average area A_m to be used:

$$A_m = A + 0.54x\Sigma y + 1.2x^2 \tag{a}$$

where A is the actual inside area. For a cube whose inside dimensions are each twice the thickness, the edge and corner terms in (a) account for 37 per cent of the whole loss. If the inside dimensions are each five times the wall thickness, this drops to 18 per cent.

3.5. Steady Flow of Heat in a Long Thin Rod. This case differs from the one in Sec. 3.2 in that losses of heat by radiation and convection are supposed to take place from the sides of the bar and must be taken into account in our calculation. To do this we must add to the Fourier equation (2.3h), written for one dimension, a term that will represent this loss of heat. Now by

* See also Sec. 11.2 and Carslaw and Jaeger.[27a, p. 366]

† For cases where the inside dimensions are less than one-fifth the wall thickness, see McAdams.[90, p. 14]

Newton's law of cooling the rate of this loss will be proportional to the excess of temperature (if not too large) of the surface element over that of the surrounding medium, which we shall assume to be at zero, and hence may be represented by b^2T where b^2 is a constant. Fourier's equation for this case then becomes

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} - b^2 T \tag{a}$$

and, when the steady state has been reached, this reduces to

$$\frac{\partial^2 T}{\partial x^2} = \frac{b^2}{\alpha} T \tag{b}$$

This is readily solved by the usual process of substituting e^{mx} for T, which gives

$$m^2 e^{mx} = \frac{b^2}{\alpha} e^{mx} \tag{c}$$

from which we get

$$m = \pm \frac{b}{\sqrt{\alpha}} \tag{d}$$

and hence

$$T = Be^{bx/\sqrt{\alpha}} + Ce^{-bx/\sqrt{\alpha}} \tag{e}$$

as the sum of two particular solutions.

3.6. The significance of the constant b is most easily shown by considering the problem entirely independently of Fourier's equation. For when the steady state has been reached in such a bar, the flow of heat per unit of time across any area of cross section A of the bar will be, at the point x,

$$-kA \frac{dT}{dx} \tag{a}$$

and, at the point $x + \Delta x$,

$$-kA \frac{d}{dx}\left(T + \frac{dT}{dx}\Delta x\right) \tag{b}$$

and consequently the excess of heat left in the bar between these two points Δx apart is

$$kA \frac{d^2T}{dx^2} \Delta x \tag{c}$$

This must escape by loss from the surface, and such loss per

unit of time will be given by $hTp\Delta x$, where h* is the so-called *surface emissivity* of the bar (see Sec. 2.5), and where $p\Delta x$ is the product of the perimeter p of the bar and the length Δx of the element, *i.e.*, the element of surface. Hence, we have

$$kA\frac{d^2T}{dx^2} = hTp \tag{d}$$

or

$$\frac{d^2T}{dx^2} = \frac{hp}{kA}T \tag{e}$$

By comparison with (3.5*b*) we then see that

$$b^2 = \frac{\alpha hp}{kA} \tag{f}$$

Writing for convenience, $hp/kA \equiv \mu^2$, our general solution (3.5*e*) takes the form

$$T = Be^{\mu x} + Ce^{-\mu x} \tag{g}$$

3.7. We may use this solution to investigate the state of temperature in a long bar, whose far end has the same temperature as the surrounding medium, while the near end is at T_1, say, the temperature of the furnace. If the area, perimeter, conductivity, and emissivity were all known or readily calculable to give μ, no further condition would be required to obtain a complete solution. In lieu of any or all of these, however, a single further condition will suffice, *i.e.*, that the point at which an intermediate temperature T_2 is reached be also known. The boundary conditions are then

$$\begin{aligned} &(1)\ T = 0 \text{ at } x = \infty \\ &(2)\ T = T_1 \text{ at } x = 0 \\ &(3)\ T = T_2 \text{ at } x = l \end{aligned} \tag{a}$$

From condition (1) we get

$$0 = Be^{\mu\infty} + Ce^{-\mu\infty} \tag{b}$$

so that

$$Be^{\infty} = 0 \quad \text{or} \quad B = 0 \tag{c}$$

Condition (2) then gives

$$T_1 = Ce^{-\mu 0} \quad \text{or} \quad C = T_1 \tag{d}$$

* For values of h, see Appendix A.

and (3) means that

$$T_2 = T_1 e^{-\mu l} \quad \text{or} \quad \mu l = \ln \frac{T_1}{T_2} \tag{e}$$

$$\therefore\ T = T_1 \left(\frac{T_1}{T_2}\right)^{-x/l} \tag{f}$$

For different bars subject to the same conditions (1) and (2) and having the same temperature T_2 at points l_1, l_2, l_3 . . . we have

$$\ln \frac{T_1}{T_2} = \mu_1 l_1 = \mu_2 l_2 = \mu_3 l_3 \cdots = \text{a constant} \tag{g}$$

which, from the definition of μ, means that

$$\frac{k_1}{l_1^2} = \frac{k_2}{l_2^2} = \frac{k_3}{l_3^2} = \cdots \frac{k_n}{l_n^2} \tag{h}$$

providing the several bars have each the same perimeter, cross section, and coefficient of emission.

3.8. This is the fundamental equation underlying the so-called *Ingen-Hausz experiment* for comparing the conductivities of different metals. The metals, in the form of rods of the same size and character of surface, are coated thinly with beeswax (melting point T_2) and are placed with one end in a bath of hot oil at temperature T_1. After standing for some time the wax is found to be melted for a certain definite distance (l) on each bar, and the conductivities are therefore in the ratio of the squares of these distances.

Another application* of (3.6g) is found in the solution for the case of the bar, heated as above, with the temperatures known at three equally spaced points.

APPLICATIONS

3.9. There could be pointed out an almost unlimited number of practical applications of these deductions for the steady flow of heat in one dimension, particularly of (3.2e), but since these are treated at length in general physics and engineering works, and especially in texts on furnaces, boilers, refrigeration, and the like, we shall be content with a few common examples.

* See Preston.[115, p. 601]

3.10. Furnace Walls. What is the loss of heat through a furnace wall 45.7 cm (18 in.) thick if the two faces are at 800°C and 60°C (1472°F and 140°F), assuming an average conductivity of 0.0024 cgs for the wall?

Here we have

$$w = \frac{0.0024 \times 740}{45.7} = 0.0389 \text{ cal/(cm}^2\text{)(sec) or 151 watts/ft}^2$$

3.11. Refrigerator or Furnace Insulation. Equation (3.4*a*) can be effectively used in studying the relation between heat gain or loss in a refrigerator or furnace, and insulation thickness. The curves of Fig. 3.2 have been calculated for the case of an

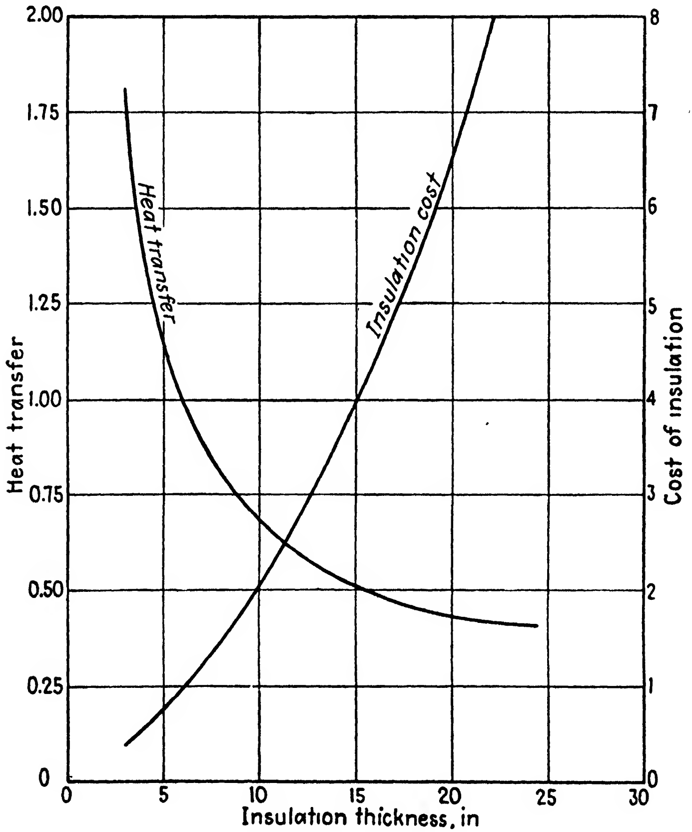

FIG. 3.2. Curves showing the relation between insulation thickness and the corresponding heat transfer and insulation cost for a rectangular refrigerator or furnace of inside dimensions 2 by 2 by 4 ft.

insulated refrigerator or frozen-food locker of inside dimensions 2 by 2 by 4 ft. They would hold equally well for a furnace of these dimensions. The heat transfer and insulation cost (*i.e.*, volume of insulation) are each taken as unity for 6 in. insulation thickness. The curves show that to reduce the heat transfer to one-half its value for 6 in. of insulation would require a thickness of 16 in., necessitating over four times the original amount of insulating material. In other words, if one were to increase materially the customary insulation thickness (4 to 6 in.) of small frozen-food lockers, the law of diminishing returns would soon come into account.

We shall make use of (3.4*a*) and (3.3*f*) in calculating the heat inflow for a frozen-food locker of inside dimensions 1.5 by 1.5 by 4 ft, with 4 in. (0.333 ft) of glass-wool insulation (k = 0.022 fph), outside of which is the box of ¾ in. (0.062 ft) thickness pine (k = 0.087 fph). The inside and outside surface temperatures will be assumed at −10°F and 70°F, respectively.

From (3.4*a*) the effective area of the insulation is

$$A_1 = 28.5 + 0.54 \times 0.333 \times 28 + 1.2 \times 0.33^2 = 33.66 \text{ ft}^2$$

Then $$R_1 = \frac{0.333}{0.022 \times 33.7} = 0.448$$

Similarly, for the box (inside dimensions 2.17 by 2.17 by 4.67 ft)

$$A_2 = 50.03 + 0.54 \times 0.062 \times 36.04 + 1.2 \times 0.062^2 = 51.25 \text{ ft}^2$$

and $$R_2 = \frac{0.062}{0.087 \times 51.2} = 0.014$$

Then $$R = R_1 + R_2 = 0.462$$

and $$q = \frac{80}{0.462} = 173 \text{ Btu/hr} = 50.7 \text{ watts}$$

Note the relatively small effect of the pine box in the matter of insulation.

3.12. Airplane-cabin Insulation. Because of the wide variation of temperature encountered by high-flying all-season planes the matter of cabin insulation may be of vital importance. The construction involves, in general, the use of two or more layers of material, with perhaps some "through metal."

Consider a cabin of cylindrical form—this can be treated as essentially a case of linear flow because of the relatively small wall thickness—with internal radius of 4 ft. Assume the wall to be 2.5 in. thick and to consist of layers as follows, starting from the inside: 0.5 in. of thickness of material of $k = 0.11$ fph; 1.8 in. of $k = 0.02$; and 0.2 in. of $k = 0.06$; with 0.1 per cent of the wall area taken up by through-metal bolts, etc., of $k = 20$. The two outer layers may be of composition sheathing material, while the center one is of glass wool or other high-grade insulator. For each foot of cabin length the average areas are $A_1 = 25.3 \text{ ft}^2$; $A_2 = 25.8 \text{ ft}^2$; $A_3 = 26.4 \text{ ft}^2$. Then, from Sec. 3.3 the individual resistances are

$$R_1 = \frac{0.042}{0.11 \times 25.3 \times 0.999} = 0.0151$$

$$R_2 = \frac{0.15}{0.02 \times 25.8 \times 0.999} = 0.291$$

$$R_3 = \frac{0.017}{0.06 \times 26.4 \times 0.999} = 0.0107$$

and $R_w = R_1 + R_2 + R_3 = 0.317$. The resistance of the through metal is $0.208/(20 \times 25.8 \times 0.001) = 0.403$. Then,

$$\frac{1}{R} = \frac{1}{0.317} + \frac{1}{0.403}$$

or $R = 0.177$.

For a 60°F temperature difference between the outside and inside surfaces the heat flow $q = 60/0.177 = 339$ Btu/hr per ft length of cabin. This means that for a 30-ft cabin the heating (or cooling) input to compensate for the cylindrical wall loss would have to be 3 kw, not allowing for windows or other openings. Contact resistance (Sec. 3.13) might diminish this somewhat but only slightly in view of the high insulating value of the central layer.

3.13. Contact Resistance. In any practical consideration of heat transfer it is disastrous to overlook the contact resistance that is offered to the heat flow by any discontinuity of material. Thus, brick masonry, as in a wall, shows a somewhat smaller conductivity than the brick itself, while powdered brick dust may have many times the insulation value of the solid material.

The thermal insulation afforded by multiple layers of paper is another illustration.

While this thermal contact resistance is not unlike its electrical analogue and in some cases might require a similar explanation, based, at least partly, on electronic considerations, it is probable that the cause in most cases lies in the intrinsic resistance of a gas-solid interface. Here we have a phenomenon, known in kinetic theory as *thermal slip*, which is really a temperature discontinuity at the gas-solid boundary and which greatly increases the resistance. This resistance varies with the gas, and Birch and Clark[16] have corrected for it in their rock conductivity determinations by making measurements with nitrogen and again with helium (which has some six times greater conductivity) as the interpenetrating gas at the rock-metal boundary.

The insulating value of porous materials has been referred to (Sec. 1.6) and explained on the basis of the low conductivity of air when in such small cells that convection is excluded. One can reason, from considerations based on thermal slip, that it should not be impossible to produce porous or cellular insulators that have lower conductivity than air itself.*

3.14. Problems

1. Compute the heat loss per day through 100 m² of brick wall ($k = 0.0020$ cgs) 30 cm thick, if the inner face is at 20°C and the outer at 0°C. How much coal must be burned to compensate this loss if the heat of combustion is 7,000 cal/gm and the efficiency of the furnace 60 per cent?

Ans. 11.5×10^7 cal; 27.4 kg

2. Calculate the rate of heat loss through a pane of glass ($k = 0.0021$ cgs) 4 mm thick and 1 m² if the two surfaces differ in temperature by 1.5°C. (NOTE: Because of the small value of h the heat transfer coefficient between glass and air, which may be of the order of only 10^{-4} cgs for still air, the difference between the two surface temperatures of the glass is much less than that of the two air temperatures.) *Ans.* 78.7 cal/sec

3. A 5-in. wall is composed of 1½ in. thickness of pine wood ($k = 0.06$ fph) on the outside and ½ in. of asbestos board ($k = 0.09$ fph) on the inside with 3 in. of mineral wool ($k = 0.024$ fph) in between. Neglecting contact resistance, calculate the rate of heat loss through the wall if the outside surface is at

* Silica aerogel is an example, although it is not certain that the cause is that indicated above.

10°F and the inside at 70°F. Also, calculate the temperature drop through each of the three layers.
Ans. 4.63 Btu/(hr)(ft²). Temperature drops: 9.6°F through the wood, 48.2°F through the mineral wool, 2.2°F through the asbestos board

4. A small electric furnace is 15 by 15 by 20 cm inside dimensions and has fire-brick ($k = 0.0021$ cgs) walls 12 cm thick. If the surface temperature of the walls is 1000°C inside and 200°C outside, what is the rate of heat loss in watts? *Ans.* 1,828 watts

5. What is the rate of heat flow in Btu/hr into a refrigerator of inside dimensions 1.5 by 2 by 3 ft with walls insulated with ground cork ($k = 0.025$ fph) 4 in. thick? Neglect the sheathing of the walls that hold the cork and assume a temperature difference of 30°F. *Ans.* 71.6 Btu/hr

6. A steam boiler with shell of ½ in. thickness evaporates water at the rate of 3.45 lb/hr per ft² of area. Assuming a heat of evaporation of 970 Btu/lb and a conductivity for the steel boiler plate of 23 fph, calculate the temperature drop through the shell. *Ans.* 6.06°F

CHAPTER 4

STEADY STATE—MORE THAN ONE DIMENSION

In this chapter we shall discuss several cases of heat flow in more than one dimension, including the important examples of spherical and cylindrical flow.

4.1. Flow of Heat in a Plane. We shall first solve Fourier's problem of the permanent state of temperatures in a thin rectangular plate of infinite length, whose surfaces are insulated. Call the width of the plate π and suppose that the two long edges are kept constantly at the temperature zero, while the one short edge, or base, is kept at temperature unity. Heat will then flow out from the base to the two sides and toward the infinitely distant end, and our problem will be to find the temperature at any point.

Take the plate as the xy plane with the base on the x axis and one side as the y axis. Then (2.3h) becomes

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \qquad (a)$$

To solve this problem, then, we must find a value for the temperature at any point that will not only be a solution of (a) but will also satisfy the boundary conditions for this case, which are

$$\begin{aligned} &(1)\ T = 0 \text{ at } x = 0 \\ &(2)\ T = 0 \text{ at } x = \pi \\ &(3)\ T = 1 \text{ at } y = 0 \\ &(4)\ T = 0 \text{ at } y = \infty \end{aligned} \qquad (b)$$

We shall attempt to find a simple particular solution of (a) that will satisfy all the conditions of (b), but, failing this, it may still be possible to combine several particular solutions, as explained in Sec. 2.4, to secure one that will do this.

4.2. Of the several ways of arriving at such a particular solution we may outline two. The first is with the aid of a device

that always succeeds when the equation is linear and homogeneous with constant coefficients. This is to assume that

$$T = e^{ay+bx} \tag{a}$$

where a and b are constants. Substituting this in (4.1a), we find at once that

$$a^2 + b^2 = 0 \tag{b}$$

which is then the condition to be satisfied in order that $T = e^{ay+bx}$ may be a solution. But this means that

$$T = e^{ay \pm axi} \tag{c}$$

for any value of a, is a solution, which is equivalent to saying

that $$T = e^{ay}e^{axi} \tag{d}$$
and $$T = e^{ay}e^{-axi} \tag{e}$$

are solutions, and by Sec. 2.4 their sum or difference divided by any constant must be a solution also. Then, since*

$$e^{i\phi} + e^{-i\phi} = 2 \cos \phi \tag{f}$$
and $$e^{i\phi} - e^{-i\phi} = 2i \sin \phi \tag{g}$$

we get, upon adding (d) and (e) and dividing by 2,

$$T = e^{ay} \cos ax \tag{h}$$

and, upon subtracting and dividing by $2i$,

$$T = e^{ay} \sin ax \tag{i}$$

Now obviously (h) does not satisfy condition (1) of (4.1b). Thus, we turn to (i), which can be seen at once to satisfy conditions (1) and (2), also (4) if a is negative. As it stands, (i) fails to meet condition (3), but it may still be possible to combine a number of particular solutions of the type of (i) that will do

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

while $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

and $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

Putting $x = i\phi$, where i is written for $\sqrt{-1}$, we see from these that $e^{i\phi} = \cos \phi + i \sin \phi$, and $e^{-i\phi} = \cos \phi - i \sin \phi$, from which ($f$) and ($g$) follow at once.

this. For if n is any positive integer,

$$T = Be^{-ny} \sin nx \tag{j}$$

fulfills the first, second, and last of the above conditions, as will also the sum

$$T = B_1 e^{-y} \sin x + B_2 e^{-2y} \sin 2x + B_3 e^{-3y} \sin 3x + \cdot \cdot \cdot \tag{k}$$

where B_1, B_2, . . . are constant coefficients. For $y = 0$ this becomes

$$T = B_1 \sin x + B_2 \sin 2x + B_3 \sin 3x + \cdot \cdot \cdot \tag{l}$$

and if it is possible to develop unity in such a series, we may still be able to satisfy condition (3) of (4.1b). Now, as we shall discuss at length in Chap. 6, Fourier showed that such a development in a trigonometric series *is* possible, the expression in this case being

$$1 = \frac{4}{\pi}\left(\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \cdot \cdot \cdot\right) \tag{m}$$

for all values of x between 0 and π. Therefore, our required solution is

$$T = \frac{4}{\pi}\left(e^{-y} \sin x + \frac{1}{3}\, e^{-3y} \sin 3x + \frac{1}{5}\, e^{-5y} \sin 5x + \cdot \cdot \cdot\right) \tag{n}$$

which satisfies (4.1a) as well as all the boundary conditions of (4.1b).

4.3. In the second method of solving (4.1a) we shall separate the variables at once by assuming that $T = XY$ where X is a function of x only, and Y of y only. Substituting, we obtain

$$Y \frac{d^2X}{dx^2} + X \frac{d^2Y}{dy^2} = 0 \tag{a}$$

or

$$\frac{1}{Y} \frac{d^2Y}{dy^2} = -\frac{1}{X} \frac{d^2X}{dx^2} \tag{b}$$

Since the two sides of this equation are functions of entirely independent variables, they can be equal only if each is equal to a constant that we may call λ^2. The solution of the partial differential equation (4.1a) is thus reduced to that of the two

ordinary differential equations

$$\frac{d^2Y}{dy^2} - \lambda^2 Y = 0 \tag{c}$$

and

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0 \tag{d}$$

These may be solved by substitutions similar to (4.2*a*) but somewhat simpler, *viz.*,

$$Y = e^{by} \quad \text{and} \quad X = e^{ax} \quad \text{respectively} \tag{e}$$

The first gives $b = \pm\lambda$; therefore,

$$Y = Be^{\lambda y} + Ce^{-\lambda y} \tag{f}$$

The second gives $a = \pm i\lambda$, so that

$$X = B'e^{i\lambda x} + C'e^{-i\lambda x} \tag{g}$$

which, from the note to Sec. 4.2, reduces, if we call

$$(B' - C')i = D$$

and $B' + C' = E$, to

$$X = D \sin \lambda x + E \cos \lambda x \tag{h}$$

Choosing $B = E = 0$ to satisfy (1), (2), and (4) of (4.1*b*), the solution resulting from the product of (*f*) and (*h*) reduces at once to (4.2*j*), and the remainder of the process is the same as before.

It may be noted that this same sort of solution will hold even if the temperature T of the base of the plate is other than unity, indeed even if it ceases to be constant and is instead a function of x, provided it can be expressed also in this latter case by a Fourier series. In case we wish to have the values of x run from 0 to l instead of from 0 to π, we must introduce as a variable the quantity $\pi x/l$, and the expressions will otherwise be the same as before. We shall discuss this at length in Chap. 6.

It is also of interest to note that our solution is entirely independent of the physical constants of the medium, so that the temperature at any point is independent of what material is used, so long as the steady state exists.

4.4. The reader who wishes to make a further study of the solution (4.2n) will find that the sum of the infinite series can be expressed in closed form to give finally

$$T = \frac{2}{\pi}\tan^{-1}\left(\frac{\sin x}{\sinh y}\right)^* \tag{a}$$

That this compact function satisfies the fundamental differential equation (4.1a) can be verified by straight forward differentiation. Obviously, it also satisfies the boundary conditions (4.1b).

This form clearly shows that x and y vary along any isotherm according to the equation

$$\frac{\sin x}{\sinh y} = \tan\frac{\pi}{2}T = \text{a constant} \tag{b}$$

By letting T take on a series of constant values from $T = 0$ to $T = 1$ in this equation, we can obtain a family of isotherms which covers the infinite plate. They all terminate at the corners ($x = 0$, $y = 0$) and ($x = \pi$, $y = 0$).

A corresponding family of lines of heat flow must everywhere be orthogonal to these isotherms as we know from Sec. 1.3. Such a family can be obtained from a function U which is conjugate to T in the analytic function $U + iT$ of $z = x + iy$, as treated in the theory of functions of a complex variable. Conjugate functions have the general property of giving orthogonal families of two-dimensional curves for constant values of the functions. The derivation of the conjugate function U from the known function T in (a) is given in Appendix L. It has the similar form

$$U = \frac{2}{\pi}\tanh^{-1}\left(\frac{\cos x}{\cosh y}\right) \tag{c}$$

Lines of heat flow in the plate then correspond to constant values of U and satisfy the equation

$$\frac{\cos x}{\cosh y} = \tanh\frac{\pi}{2}U = \text{a constant} \tag{d}$$

It is obvious that the line of heat flow for $U = 0$ is a straight line parallel to the y axis at $x = \pi/2$, *i.e.*, along the center line

* See Byerly[23] Art. 58.

of the plate, parallel to the two sides. This checks with the physical symmetry of the external temperatures.

If x is allowed to extend indefinitely in both directions, the above solution corresponds to the physical case in which T on the boundary $y = 0$ is kept alternately equal to $+1$ and -1 over ranges of $x = \pi$.

Problem 1 of Sec. 4.12 calls for a graph of the case considered in these last four sections, while in Secs. 11.2 to 11.5 there are a number of other isotherm and flow-line diagrams.

4.5. Flow of Heat in a Sphere. To investigate the radial flow of heat in a sphere, we must first replace the rectilinear coordinates, x, y, and z in (2.3h) by the single variable r. This is done by means of the following transformation:

$$\frac{\partial T}{\partial x} = \frac{dT}{dr}\frac{\partial r}{\partial x} = \frac{dT}{dr}\frac{x}{r} \tag{a}$$

because, since $r^2 = x^2 + y^2 + z^2, \qquad \dfrac{\partial r}{\partial x} = \dfrac{x}{r}$

also

$$\frac{\partial^2 T}{\partial x^2} = \frac{d^2T}{dr^2}\frac{x^2}{r^2} + \frac{dT}{dr}\frac{1}{r} - \frac{dT}{dr}\frac{x^2}{r^3} \tag{b}$$

with similar expressions for the derivatives with respect to y and z. We thus obtain

$$\nabla^2 T \equiv \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{d^2T}{dr^2} + \frac{2}{r}\frac{dT}{dr} \tag{c}$$

Since, however,

$$\frac{d(rT)}{dr} = r\frac{dT}{dr} + T \quad \text{and} \quad \frac{d^2(rT)}{dr^2} = r\frac{d^2T}{dr^2} + 2\frac{dT}{dr} \tag{d}$$

we have

$$\nabla^2 T = \frac{1}{r}\frac{d^2(rT)}{dr^2} \tag{e}$$

The Fourier equation for steady radial heat flow thus becomes

$$\frac{1}{r}\frac{d^2(rT)}{dr^2} = 0 \tag{f}$$

and its integral may at once be written

$$T = B + \frac{C}{r} \tag{g}$$

For boundary conditions we may take

$$\begin{aligned} &(1)\ T = T_1 \text{ at } r = r_1 \\ &(2)\ T = T_2 \text{ at } r = r_2 \end{aligned} \tag{h}$$

where r_1 and r_2 are, respectively, the internal and external radii of the hollow sphere. These conditions give, on substitution in (g), after the elimination of B and C,

$$T = \frac{r_2T_2 - r_1T_1}{r_2 - r_1} + \frac{r_1r_2(T_1 - T_2)}{r(r_2 - r_1)} \tag{i}$$

This expresses the temperature for any point of the sphere and also shows that the isothermal surfaces are concentric spheres. The rate of flow of heat per unit area in the direction r is given by

$$w = -k\frac{dT}{dr} = \frac{k(T_1 - T_2)r_1r_2}{r^2(r_2 - r_1)} \tag{j}$$

and the total quantity that flows out in unit time is

$$q = 4\pi r^2 w = \frac{4\pi k(T_1 - T_2)r_1r_2}{r_2 - r_1} \tag{k}$$

If q* units of heat are released per unit of time at a point (*i.e.*, in a region of small spherical volume) in an infinite medium, at zero initial temperature, the steady state of the temperature in the medium can be calculated at once from (g), (h), and (k). Boundary condition (2) of (h) becomes $T = 0$ at $r = \infty$; thus (g) becomes

$$T = T_1\frac{r_1}{r} \tag{l}$$

We can get T_1 from (k) by writing

$$4\pi r^2 w = q; \qquad T_2 = 0; \qquad r_2 = \infty \tag{m}$$

Thus,

$$q = 4\pi k T_1 r_1 \tag{n}$$

or

$$T_1 = \frac{q}{4\pi k r_1} \tag{o}$$

Then,

$$T = \frac{q}{4\pi k r} \tag{p}$$

Compare this with (9.5m).

* In Chaps. 8 and 9 the symbol Q' is, in general, used for the rate of heat generation.

4.6. Radial Flow of Heat in a Cylinder. Let the axis of the cylinder be the z axis. Then, the problem is similar to that for the sphere, save that now we are concerned with only two dimensions and may put $r^2 = x^2 + y^2$. By a process similar to that by which (4.5c) and (4.5e) were obtained we then get

$$\nabla^2 T = \frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} = \frac{1}{r}\frac{d(rdT/dr)}{dr} = 0 \qquad (a)$$

The integral of this is

$$T = B \ln r + C \qquad (b)$$

which gives, by the use of boundary conditions quite similar to those of (4.5h),

$$T_1 = B \ln r_1 + C; \qquad T_2 = B \ln r_2 + C \qquad (c)$$

and from these we obtain

$$T = \frac{(T_1 - T_2) \ln r}{\ln r_1 - \ln r_2} + \frac{T_1 \ln r_2 - T_2 \ln r_1}{\ln r_2 - \ln r_1} \qquad (d)$$

The rate of flow per unit area is given by

$$w = \frac{k(T_1 - T_2)}{r(\ln r_2 - \ln r_1)} \qquad (e)$$

and the quantity of heat that flows out through unit length of the cylinder per second by

$$2\pi r w = \frac{2\pi k(T_1 - T_2)}{\ln r_2 - \ln r_1} \qquad (f)$$

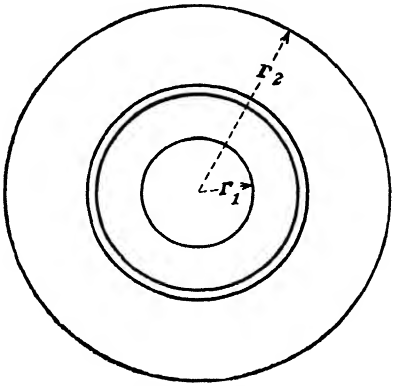

FIG. 4.1. Section of a sphere or cylinder.

4.7. The results of the two preceding sections may be very simply obtained from the linear-flow equation, for the flow in any element of small angle is essentially in one direction. However, the cross-sectional area is continually increasing, being obviously proportional to the distance from the center in the cylindrical case and to the square of this distance in the spherical. From (1.3c) we get at once as the rate of flow q through any spherical shell of area $4\pi r^2$ and thickness dr,

$$q = -k4\pi r^2 \frac{dT}{dr} \tag{a}$$

Writing this as

$$dT = -\frac{q\,dr}{4\pi k r^2} \tag{b}$$

we have, on integration,

$$T_1 - T_2 = \frac{q}{4\pi k}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \tag{c}$$

or

$$q = \frac{4\pi k(T_1 - T_2)r_1 r_2}{r_2 - r_1} \tag{d}$$

which is identical with (4.5*k*).

Similarly, for unit length of a cylinder,

$$q = -k2\pi r \frac{dT}{dr} \tag{e}$$

or

$$dT = -\frac{q\,dr}{2\pi k r} \tag{f}$$

which gives, on integration,

$$T_1 - T_2 = \frac{q}{2\pi k} \ln\left(\frac{r_2}{r_1}\right) \tag{g}$$

or

$$q = \frac{2\pi k(T_1 - T_2)}{\ln (r_2/r_1)} \tag{h}$$

which is essentially the same as (4.6*f*). By integrating (*b*) and (*f*) between T_1 (or T_2) and T, and correspondingly between r_1 (or r_2) and r, we can obtain at once (4.5*i*) and (4.6*d*) on substituting values of q from (*d*) and (*h*).

Carrying a step further our treatment of spherical and cylindrical flow with the aid of the fundamental linear-flow equation, we may write from (*d*),

$$q = \frac{4\pi k(T_1 - T_2)r_1 r_2}{r_2 - r_1} = kA_m \frac{T_1 - T_2}{r_2 - r_1} \tag{i}$$

where A_m is the mean value of the area to be used in the spherical case. This gives

$$A_m = 4\pi r_1 r_2 = \sqrt{A_1 A_2} \tag{j}$$

which means that the average area to be taken if we use the simple linear-flow equation for the hollow sphere is the geometric mean of the inner and outer surfaces.

For a cylinder of length L,

$$q = \frac{2\pi k(T_1 - T_2)L}{\ln (r_2/r_1)} = kA_m \frac{T_1 - T_2}{r_2 - r_1} \tag{k}$$

$$\text{or} \quad A_m = \frac{2\pi r_2 L - 2\pi r_1 L}{\ln (2\pi r_2 L/2\pi r_1 L)} = \frac{A_2 - A_1}{\ln (A_2/A_1)}$$

$$= \frac{A_2 - A_1}{2.303 \log_{10} (A_2/A_1)} \tag{l}$$

If A_1 and A_2 are not far different, we can frequently use the arithmetic mean value for A_m instead of the logarithmic mean as given by (*l*) and still keep within prescribed limits of error. Thus, if $A_2/A_1 = 2$, the arithmetic mean is only 4 per cent larger than the logarithmic; while if A_2/A_1 does not exceed 1.4, the difference is less than 1 per cent.

Thermal-resistance equations, in particular (3.3*f*), may be applied to a series of concentric spherical or cylindrical shells if the areas A_a, A_b, etc., of (3.3*f*) are evaluated from (*j*) or (*l*).

APPLICATIONS

4.8. Covered Steam Pipes. Some of the best applications of the theory of Secs. 4.5 and 4.6 are the various radial-flow methods of measuring thermal conductivity described in Sec. 12.5. We shall confine ourselves here, however, to applications of a slightly different sort. As an example of the use of (4.6*f*) let us investigate the heat loss per unit length of a 2-in. steam pipe (outside diameter 2.375 in. or 6.04 cm), protected by a covering 1 in. (2.54 cm) thick of conductivity 0.0378 fph (0.000156 cgs). Assume the inner surface of the covering to be at the pipe temperature of 365°F (185°C) and the outer at 117°F (47.2°C).

Then from (4.6*f*)

$$2\pi r w = \frac{2\pi \times 0.0378 \times 248}{2.303 \log_{10} 1.84}$$

$$= 96.6 \text{ Btu/hr per ft of pipe length}$$

$$= 0.222 \text{ cal/sec per cm length}$$

It is of interest to note that double this thickness of covering would still allow a loss of 59.8 Btu/hr per ft length for the same temperature range, or only 38 per cent decrease in loss for 158 per cent added covering material. That the proportional sav-

ing* is greater for a larger pipe is shown by the curves of Fig. 4.2.

The temperature of current-carrying wires as affected by the insulation is also a question that might be studied with the

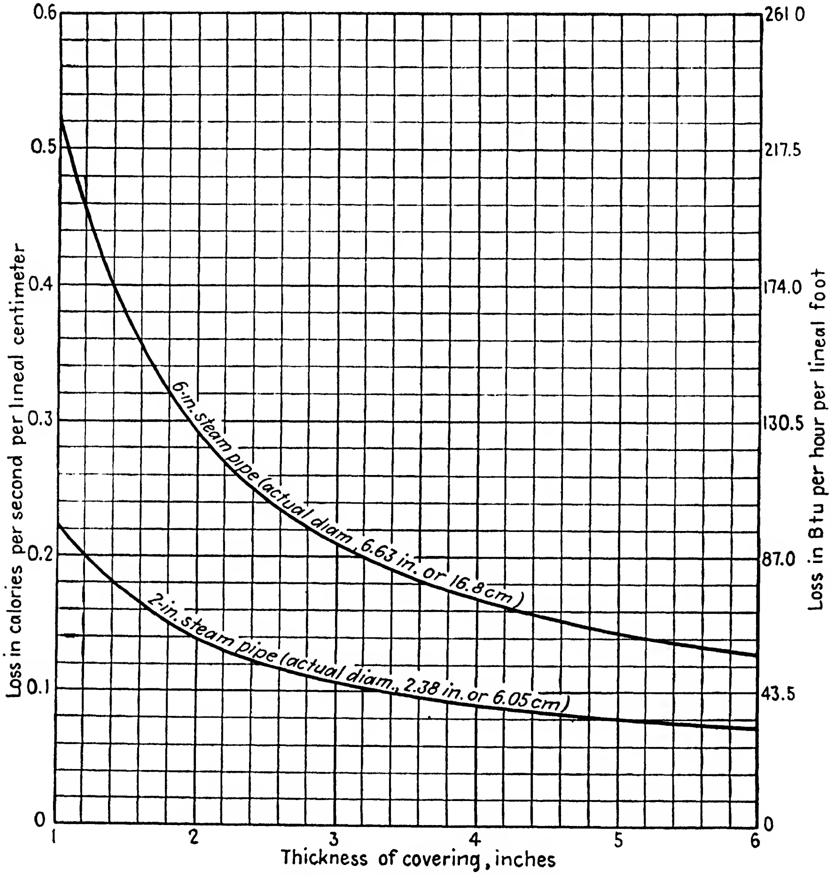

FIG. 4.2. Curves showing the relation of heat loss to thickness of covering, for two sizes of steam pipe, with temperature drop through the covering of 248°F or 138°C. Conductivity of covering, 0.0378 fph or 0.000156 cgs.

aid of the foregoing equations. It can easily be shown that a wire insulated with a covering of not too low thermal conductivity may run cooler, for a given current, than the same wire

* For a discussion of the most economical thickness for pipe coverings see Walker, Lewis, and McAdams.[157, p 126]

if bare; the insulation in this case produces, effectively, so much more cooling surface. A similar case for steam pipes would occur under special circumstances of small pipe and very poor insulation.

4.9. Flow of Heat in Solid and Hollow Cones. A truncated solid cone of not too large angle is in effect part of a hollow sphere, the fraction being the ratio of its solid angle to 4π. The rate of flow down such a cone may be determined at once from (4.7*i*). The hollow cone, if of uniform thickness, is made from the sector of a circle. The heat flow may be found with the aid of (4.7*l*), using for A_2 and A_1 the sectional areas (metal only) for the large and small ends. A hollow cone is frequently used to connect the outlet pipe of a vessel (Fig. 4.3) containing very hot or very cold liquids with a base or surface at room temperature. Assume that such a cone of metal of low conductivity (*e.g.*, "inconel"; $k = 0.036$ cgs) 0.5 mm in thickness connects a pipe of 3 cm diameter with the exterior metal sheath of the insulated vessel, the base of the cone being 10 cm in diameter and its length, measured along the cone, 12 cm. If the pipe is at 200°C and the base of the cone at 0°C, what is the rate of heat loss through the cone?

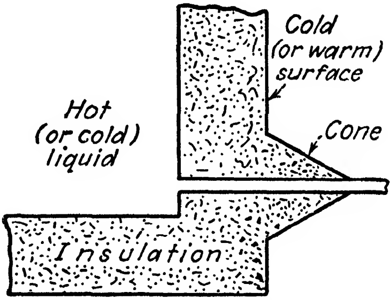

FIG. 4.3. Hollow cone used in connection with insulated vessel

Such a cone is equivalent to a sector of a circle with

$$r_2 - r_1 = 12 \text{ cm}$$

If p represents its fraction of a circle, $2\pi r_2 p = \pi \times 10$ and $2\pi r_1 p = \pi \times 3$. From these relations we find at once $r_1 = 5.14$ cm; $r_2 = 17.14$ cm; $p = 0.292$. From (4.6*f*) we then have as the flow of heat down the cone

$$q = 2\pi r p w \times 0.05 = \frac{2\pi \times 0.292 \times 0.036 \times 200 \times 0.05}{2.303 \log_{10} (17.14/5.14)}$$

$$= 0.55 \text{ cal/sec} \qquad (a)$$

If the pipe is directly connected with the outer sheath as the

center of a 10-cm diameter circle of this same metal 0.5 mm thick, and if it is assumed that the circumference of this circle is at 0°C as was the case for the cone, the loss will now be

$$q = 2\pi rw = \frac{2\pi \times 0.036 \times 200 \times 0.05}{2.303 \log_{10} {}^{10}\!/_{3}} = 1.88 \text{ cal/sec} \qquad (b)$$

It is evident that the cone lessens the heat waste, the ratio of the losses under these conditions being the fraction p.

4.10. Subterranean Temperature Sinks and Power-development; Geysers. The question is sometimes raised as to the possibility of power development from large areas of heated rock, *e.g.*, old lava beds, etc. Its answer forms an interesting application of (4.5k) and (4.5p). Assume that an old buried lava bed ($k = 1.2$ fph) at temperature 500°F has a deep hole ending in a spherical cavity of 4 ft radius. Water is fed into this and the resultant steam used for power purposes. When a steady state has been reached, what steady power development might be expected? Assume that the temperature of the interior of the cavity must not fall below 300°F.

We shall treat this problem as a point sink (negative source) and consider temperature conditions at $r = 4$ ft where the temperature is 200°F below that of the lava. We may then use (4.5p) with the understanding that we are not concerned with the temperature distribution inside $r = 4$ ft providing that the temperature for this radius is kept steadily 200° below the initial value.

$$\text{Then} \qquad 200 = \frac{q}{4\pi \times 1.2 \times 4} \quad \text{or} \quad q = 12{,}050 \text{ Btu/hr} \qquad (a)$$

This means that only 4.73 hp could be developed. Conditions while the steady state is being approached, and the time involved in reaching the steady state, will be studied later (Secs. 9.4 and 9.10).

It is evident that these same principles would apply to a study of geysers if conditions are such that the heat is supplied at or near the bottom of the tube. In general, however, the inflow of heat is probably along a considerable length of tube, and accordingly it is a case of cylindrical rather than spherical flow. We shall treat this case in Sec. 9.10.

4.11. Gas-turbine Cooling. A major problem in gas-turbine design is that of keeping the temperatures of the parts from running too high. The cooling of the rotor is principally due to gas convection, but it is important to know how large a part conduction cooling may play. It is possible to make a simple approximate calculation of the conduction cooling, assuming that the heat flows radially in from the bladed periphery of the rotor disk and is carried away at the center by conduction along the axle—or perhaps by liquid cooling in the axle.

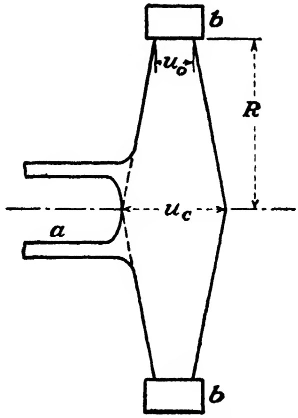

FIG. 4.4. Section of gas-turbine rotor: (*a*) hollow axle, (*b*) blading.

Such a rotor is shown in section in Fig. 4.4. Let u_c be the thickness of the disk at the center and $u_c - pr$ the thickness at radius r, where $p = (u_c - u_0)/R$, u_0 being the thickness where the blading begins and R the corresponding radius. From (3.3*g*),

$$q = \frac{k(T_2 - T_1)}{\displaystyle\int_{r_1}^{r_2} \frac{dr}{2\pi r(u_c - pr)}} \tag{a}$$

But since (Appendix B)

$$\int \frac{dx}{x(a + bx)} = \frac{1}{a} \ln \frac{x}{a + bx} \tag{b}$$

we have

$$\int_{r_1}^{r_2} \frac{dr}{r(u_c - pr)} = \frac{1}{u_c} \ln \frac{r_2(u_c - pr_1)}{r_1(u_c - pr_2)} \tag{c}$$

Then

$$q = \frac{2\pi u_c k(T_2 - T_1)}{2.303 \log_{10} \dfrac{r_2(u_c - pr_1)}{r_1(u_c - pr_2)}} \tag{d}$$

Note that for a disk of uniform unit thickness, (*d*) reduces to (4.6*f*) or (4.7*h*), as it should.

We shall calculate the rate of radial heat flow from blading to center for a turbine rotor of dimensions $R = 25$ cm (9.84 in.); $u_0 = 2$ cm (0.79 in.); $u_c = 7$ cm (2.76 in.). Assume the material of conductivity 0.09 cgs (22 fph) for the average temperatures involved, and take the temperatures as 600°C (1112°F) at $r_2 = R = 25$ cm, and 320°C (608°F) at $r_1 = 5$ cm (1.97 in.).

Then from (d) we calculate the rate of heat flow from periphery to center as $q = 409$ cal/sec $= 5846$ Btu/hr. For a disk of 2 cm uniform thickness we can calculate from (4.6f) or (4.7h) that, for the same temperatures as used above,

$$q = 197 \text{ cal/sec} = 2810 \text{ Btu/hr}$$

The smallness of these figures shows clearly the inadequacy of conduction cooling alone.

It is evident at once that, having calculated q for temperatures T_1 and T_2, (d) can be used to find the temperature for any other radius of the disk, assuming conduction cooling alone as operative.

4.12. Problems

1. Plot the temperatures for a dozen points in a plane such as is treated in Secs. 4.1 to 4.4, and draw the isotherms and lines of flow.

2. A wire whose resistance per cm length is 0.1 ohm is embedded along the axis of a cylindrical cement tube of radii 0.05 cm and 1.0 cm. A current of 5 amp is found to keep a steady difference of 125°C between the inner and outer surfaces. What is the conductivity of the cement and how much heat must be supplied per cm length? *Ans.* 0.0023 cgs; 0.597 cal/sec

3. A hollow lead ($k = 0.083$ cgs) sphere whose inner and outer diameters are 1 cm and 10 cm is heated electrically with the aid of a 10-ohm coil placed in the cavity. What current will keep the two surfaces at a steady difference of temperature of 5°C? Also, at what rate must heat be supplied?

Ans. 1.10 amp; 2.90 cal/sec

4. Calculate the rate of heat loss from a 10-in. (actual diameter 10.75 in.) steam pipe protected with a 2-in. covering of conductivity 0.04 fph if the inner and outer surfaces of the covering are at 410°F and 90°F.

Ans. 254 Btu/hr per ft length

5. A 60-watt lamp is buried in soil ($k = 0.002$ cgs) at 0°C and burned until a steady state of temperature is reached. What is the temperature 30 cm away? *Ans.* 19°C

6. Calculate the rate of heat flow for the following cases, the metal being nickel ($k = 0.142$ cgs) with surfaces insulated: (a) a circular disk 1 mm thick and 10 cm in diameter with a central hole 1 cm in diameter and with 100°C temperature difference between hole and edge; (b) a cone of the same thickness of sheet nickel, 20 cm long, 1 cm mean diameter at the small end, and 10 cm diameter at the large, and with 100°C temperature difference between the ends; (c) a solid cone* of the same dimensions and same temperature difference. Measure cone lengths on the element.

Ans. 3.87 cal/sec; 0.87 cal/sec; 5.65 cal/sec

* It can be readily shown that a cone of half angle θ has a solid angle of $2\pi(1 - \cos\theta)$.

CHAPTER 5

PERIODIC FLOW OF HEAT IN ONE DIMENSION

5.1. We shall now take up the problem of the flow of heat in one dimension that takes place in a medium when the boundary plane, normal to the direction of flow, undergoes simply periodic variations in temperature. This problem occupies in a way an intermediate place between those of the steady state already considered and the more general cases that can be treated only after a familiarity has been gained with Fourier's series; for in the former cases the temperature at any point has been constant, while in the latter it is a more or less complicated function of the time, rarely reaching the same value twice at a given point; but in the present case the temperature at each point in the medium varies in a simply periodic manner with the time, and while the temperature condition is by no means "steady," as we have defined this term, it duplicates itself in each complete period.

The problem derives its interest and importance from its very practical applications. The surface of the earth undergoes daily and annual changes of temperature that are nearly simply periodic, and it is frequently desirable to know at just what time a maximum or minimum of temperature will be reached at any point below the surface, as well as the actual value of this temperature. Such knowledge would be of value, *e.g.*, in determining the necessary depth for water pipes, to avoid danger of freezing, or in giving warning of just when to anticipate such danger after the appearance of a "cold wave," *i.e.*, one of those roughly periodic variations of temperature that frequently characterize a winter.

5.2. Solution. Our fundamental equation for this case is the Fourier conduction equation

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T \qquad (a)$$

written in one dimension

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \tag{b}$$

and the solution must fit the boundary condition

$$T = T_0 \sin \omega t \qquad \text{at } x = 0 \tag{c}$$

As the equation (*b*) is linear and homogeneous with constant coefficients, we can arrive at a particular solution by the same device used in Sec. 4.2, *viz.*, by the assumption that

$$T = Be^{bt+cx} \tag{d}$$

where b and c are constants. Substitution in (*b*) shows that this is a solution, provided only that

$$b = \alpha c^2 \tag{e}$$

Thus, we have as a solution

$$T = B \exp\left(bt \pm x\sqrt{\frac{b}{\alpha}}\right) \tag{f}$$

If b is replaced by $\pm i\gamma$, this becomes

$$T = B \exp\left(\pm i\gamma t \pm x\sqrt{\frac{\gamma}{\alpha}}\sqrt{\pm i}\right) \tag{g}$$

But
$$\sqrt{i} = \pm \frac{(1+i)^*}{\sqrt{2}} \tag{h}$$

and
$$\sqrt{-i} = \pm \frac{(1-i)}{\sqrt{2}} \tag{i}$$

so that (*g*) becomes

$$T = B \exp\left[\pm i\gamma t \pm x\sqrt{\frac{\gamma}{2\alpha}}(1 \pm i)\right] \tag{j}$$

or
$$T = B \exp\left(\pm x\sqrt{\frac{\gamma}{2\alpha}}\right) \exp\left[\pm i\left(\gamma t \pm x\sqrt{\frac{\gamma}{2\alpha}}\right)\right] \tag{k}$$

From the several solutions contained in (*k*) other particular solutions may be built up by addition, such as

$$T = B \exp\left(-x\sqrt{\frac{\gamma}{2\alpha}}\right)\left\{\exp\left[i\left(\gamma t - x\sqrt{\frac{\gamma}{2\alpha}}\right)\right] - \exp\left[-i\left(\gamma t - x\sqrt{\frac{\gamma}{2\alpha}}\right)\right]\right\} \tag{l}$$

* $(1+i)^2 = 1 + 2i - 1 = 2i \qquad \therefore \sqrt{i} = \pm \dfrac{(1+i)}{\sqrt{2}}$

and from Sec. 4.2 this may be written

$$T = Ce^{-x\sqrt{\gamma/2\alpha}} \sin\left(\gamma t - x\sqrt{\frac{\gamma}{2\alpha}}\right) \tag{m}$$

Other solutions may be formed in the same way, care being taken to note, however, that, from the manner of its formation [see (*f*)], the sign before i in each term of (*j*) must be the same. This will be found equivalent to saying that the same sign must be used before $x\sqrt{\gamma/2\alpha}$ in each term of equations like (*l*). With this in mind we may write at once as other particular solutions

$$T = C'e^{x\sqrt{\gamma/2\alpha}} \sin\left(\gamma t + x\sqrt{\frac{\gamma}{2\alpha}}\right) \tag{n}$$

$$T = De^{-x\sqrt{\gamma/2\alpha}} \cos\left(\gamma t - x\sqrt{\frac{\gamma}{2\alpha}}\right) \tag{o}$$

and

$$T = D'e^{x\sqrt{\gamma/2\alpha}} \cos\left(\gamma t + x\sqrt{\frac{\gamma}{2\alpha}}\right) \tag{p}$$

Of these four solutions, (*n*) and (*p*) demand that the temperature increase indefinitely as x increases, which is evidently absurd, while (*o*) is excluded by (*c*). Equation (*m*) will satisfy this condition if C is put equal to T_0 and γ to ω. Making these changes, we then have as the solution

$$T = T_0e^{-x\sqrt{\omega/2\alpha}} \sin\left(\omega t - x\sqrt{\frac{\omega}{2\alpha}}\right) \tag{q}$$

which expresses the temperature at any time t at any distance x from the surface.

5.3. Amplitude, Range. The equation (5.2*q*) is that of a wave motion whose rapidly decreasing amplitude is given by the factor $T_0e^{-x\sqrt{\omega/2\alpha}}$. The range of temperature, or maximum variation, for any point below the surface is given by

$$T_R = 2T_0e^{-x\sqrt{\omega/2\alpha}} = 2T_0e^{-x\sqrt{\pi/\alpha P}} \tag{a}$$

putting for ω its value $2\pi/P$, where P is the period. T_0 is the amplitude, or half range, at the surface. This shows at once that the slower the variation of temperature the greater the range in the interior of the body.

5.4. Lag, Velocity, Wave Length. The time at which a maximum or minimum of temperature will occur at any point is evidently that for which

$$\omega t - x\sqrt{\frac{\omega}{2\alpha}} = (2n+1)\frac{\pi}{2} \tag{a}$$

or

$$t = \frac{x\sqrt{\omega/2\alpha} + (2n+1)\pi/2}{\omega} \tag{b}$$

odd values of n giving minima, and even, maxima. Fixing our attention on the minimum that occurs at the surface when, say, $\omega t = 3\pi/2$, we see that if x and t are both supposed to increase so that

$$\omega t - x\sqrt{\frac{\omega}{2\alpha}} = \frac{3\pi}{2} \tag{c}$$

we may think of this particular minimum being propagated into the medium and reaching any point x at the time given by this equation. This is later than its occurrence at the surface by an amount

$$t_1 = x\sqrt{\frac{1}{2\alpha\omega}} = \frac{x}{2}\sqrt{\frac{P}{\pi\alpha}} \tag{d}$$

which may be called the "lag" of the temperature wave. The same reasoning holds for the maximum, or zero, or any other phase.

The apparent velocity of such a wave in the medium is given from (d) by

$$V = \frac{x}{t_1} = 2\sqrt{\frac{\pi\alpha}{P}} \tag{e}$$

but this is merely the rate at which a given maximum or minimum may be said to travel and has nothing to do with the actual speed with which the heat energy is transmitted from particle to particle.

From (e) we may deduce as the expression for the wave length of such a wave

$$\lambda = VP = 2\sqrt{\pi\alpha P} \tag{f}$$

Equations (d) to (f) may be used to measure the diffusivity

of any medium from determinations of the lag, velocity, or wave length.

5.5. Temperature Curve in the Medium. The form of this curve at any given time may be conveniently investigated by differentiating (5.2q) with respect to x to find the maxima and minima of the curve, which, of course, will be distinguished from the maxima and minima above treated. Then, writing

$$\mu \equiv \sqrt{\frac{\omega}{2\alpha}}$$

we have

$$\tan(\omega t - \mu x) = -1 \tag{a}$$

or

$$x = \frac{\pi/4 + \omega t}{\mu}, \frac{5\pi/4 + \omega t}{\mu}, \frac{9\pi/4 + \omega t}{\mu}, \ldots \tag{b}$$

This shows that the minima and maxima are equally spaced, and if we note that the corresponding minima and maxima of the pure sine curve

$$y = \sin(\omega t - \mu x) \tag{c}$$

occur at

$$x = \frac{\pi/2 + \omega t}{\mu}, \frac{3\pi/2 + \omega t}{\mu}, \ldots \tag{d}$$

they are seen to be nearer the surface than these latter by an amount $\pi/4\mu$. This means that, when $t = nP$ [or $(n + \frac{1}{2})P$], the first minimum (or maximum) is found at just half the distance of the corresponding minimum (or maximum) for the sine curve. This is illustrated in the solid line curve in Fig. 5.1, which gives the temperatures for different depths for the diurnal wave in soil of diffusivity = 0.0049 cgs. The broken line is the curve of amplitudes for an amplitude, or half range, of 5° at the surface.

5.6. Flow of Heat per Cycle through the Surface. This is readily computed by forming the temperature gradient from (5.2q) and then integrating it over a half period in which the gradient is of one sign, *i.e.*, going from zero to zero. Thus,

$$\frac{\partial T}{\partial x} = T_0 e^{-x\sqrt{\omega/2\alpha}} \left(-\sqrt{\frac{\omega}{2\alpha}}\right) \left[\sin\left(\omega t - x\sqrt{\frac{\omega}{2\alpha}}\right) + \cos\left(\omega t - x\sqrt{\frac{\omega}{2\alpha}}\right)\right] \tag{a}$$

and

$$\frac{Q}{A} = -k \int_{-P/8}^{3P/8} \left(\frac{\partial T}{\partial x}\right)_{x=0} dt = -k \int_{-\pi/4\omega}^{3\pi/4\omega} \left(\frac{\partial T}{\partial x}\right)_{x=0} dt$$
$$= kT_0 \sqrt{\frac{2P}{\pi\alpha}} \text{ cal/cm}^2\text{, or Btu/ft}^2 \quad (b)$$

The limits of integration in (*b*) are determined by the fact that $\partial T/\partial x$ is not in phase with T but, for $x = 0$, has a minimum at $t = P/8 = \pi/4\omega$ and is zero at $t = -P/8 = -\pi/4\omega$ and $t = 3P/8 = 3\pi/4\omega$. The amount of heat given by (*b*) flows through the surface into the material during one half the cycle in which $\partial T/\partial x$ is negative and out again during the other half.

APPLICATIONS

5.7. With the aid of the foregoing equations we may investigate the penetration of periodic temperature waves into the earth. The questions of interest and importance in this connection are (1) the range or variation of temperature at various depths for the diurnal and annual changes; and (2) the velocity of penetration of such waves, and hence the time at which the maximum or minimum may be expected to occur at various depths.

5.8. Diurnal Wave. First consider the diurnal or daily wave. If the surface of the soil varies daily, at a certain season, from +16 to −4°C (60.8 to 24.8°F), what is the range at 30 cm (11.8 in.) and 1 m (39.4 in.)? The mean of the above temperatures is +6°C; and as condition (5.2*c*) supposes a mean temperature of zero, our temperature scale must be reduced by the subtraction of 6°, which can be added again later if necessary. In this case, then, T_0 is 10°C and P = 86,400 sec. Using the constants for ordinary moist soil (α = 0.0049 cgs), (5.3*a*) shows that the range is reduced from 20° at the surface to only 0.07 of this, or 1.4°C (2.5°F), at 30 cm below, and to less than 0.004°C at 1 m below. Since a range of 12° would just be sufficient in this case—assuming an average temperature of 6°C in the soil—to reach a freezing temperature, we conclude that a layer of soil 6 cm thick will be enough to prevent freezing under these conditions. Dry soil will afford even smaller penetration than

this, and in the damp soil we have neglected the latent heat of freezing of the soil, which, while nearly negligible for small water content, would still reduce the penetration of the freezing temperature somewhat. We may also deduce from (5.4*d*) that the maximum or minimum temperature at 30 cm would lag

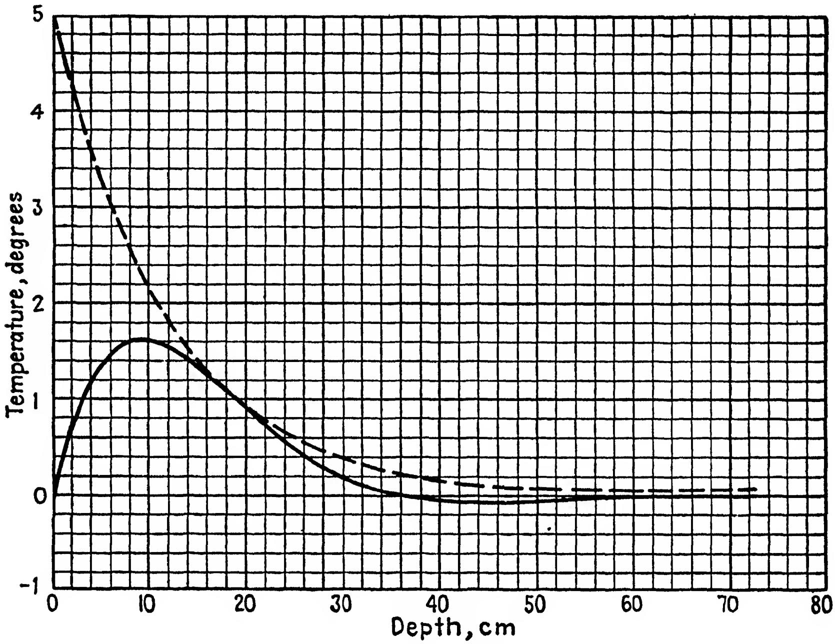

FIG. 5.1. Curves showing the penetration of the diurnal temperature wave in soil of diffusivity 0.0049 cgs. Solid line is curve of temperatures at time $t = (n + \frac{1}{2})P$ (*i.e.*, in the early evening). Broken line is curve of amplitudes for an amplitude, or half range, of 5° at the surface.

some 35,000 sec, or 9.7 hr, behind that at the surface. In a series of soil temperature measurements by MacDougal[92] the lag of the maximum at 30 cm depth was found to be from 8 to 12 hr, and the range generally less than one-tenth of the range in air, both figures being in substantial agreement with the above deductions.

5.9. Annual Wave. For the annual wave the variation for temperate latitudes may be taken as 22 to −8°C (71.6 to 17.6°F). The range at 1 m will then be reduced to 19°C, while at 10 m below the surface it will be only 0.33°C. The freezing

temperature will penetrate to a depth of less than 170 cm (5.6 ft).* From (5.4e) the velocity of penetration of such a wave is 0.000045 cm/sec, or 3.9 cm per day. For a soil of this diffusivity, then, the minimum temperature at a depth of about 7 m (23 ft) would occur in July and the maximum in January.

Table 5.1 is compiled from measurements of underground temperatures in Japan, cited by Tamura.[144] The computed temperature range and lag were calculated for a diffusivity of $\alpha = 0.0027$ cgs by (5.3a) and (5.4d).

TABLE 5.1

Depth, cm	Observed annual range, °C	Calculated annual range, °C	Observed lag, days	Calculated lag, days
0	28.2	28.2	0	0
30	22.7	23.4	2.5	10.6
60	18.7	19.5	9.0	21.6
120	14.0	13.5	35.0	42.3
300	5.2	4.6	93.5	106.0
500	1.3	1.3	177.5	176.5
700	0.4	0.4	267.0	247.0

Fitton and Brooks[40] have published a series of soil temperatures in the United States† that give much material for calculations on lag, range, diffusivity, etc. Thus, a series of measurements at Bozeman, Mont., at depths from 1 to 10 ft give an annual temperature range at the greater depth of only 0.416 that at the shallower and a lag for the greater depth of 55 days behind the other. Using (5.3a), we have

$$0.416 = \exp\left[-(x_2 - x_1)\sqrt{\frac{\pi}{\alpha P}}\right] \qquad (a)$$

and, putting $x_2 - x_1 = 9$ ft $= 274$ cm and

$$P = 1 \text{ year} = 3.156 \times 10^7 \text{ sec}$$

we get $\alpha = 0.0097$ cgs, a high value for soil. Computing from

* In reality, considerably less than this, because of the latent heat of freezing.

† See also Smith.[134]

the lag with the aid of (5.4*d*), we use

$$t_2 - t_1 = 55 \text{ days} = 4.75 \times 10^6 \text{ sec} = \frac{274}{2} \sqrt{\frac{P}{\pi\alpha}} \qquad (b)$$

from which we get $\alpha = 0.0083$ cgs. Similarly, in sandy loam at New Haven, Conn., a series of readings at depths from 3 to 12 in. give an average daily range at the former depth 7.5 times that at the latter. From (5.3*a*) we then have

$$7.5 = \exp\left(22.9\sqrt{\frac{\pi}{\alpha P}}\right) \qquad (c)$$

where $P = 86{,}400$ sec. This gives $\alpha = 0.0047$ cgs.

Birge, Juday, and March[17] have made a study of the temperatures in the mud at the bottom of a lake (Mendota) by means of a special resistance thermometer that could be driven into the mud to a depth of 5 m. From a large series of measurements the amplitude and lag of the annual temperature wave could be determined. This allowed the computation of the diffusivity of the mud and (with auxiliary data) of the annual heat flow [see (5.6*b*)] into and out of the lake through the bottom.

5.10. Cold Waves. While the preceding formulas were developed on the assumption of a simply periodic temperature wave that continues indefinitely, they are still applicable with a fair degree of approximation to temporary variations of a roughly periodic nature, such as cold waves. A good example of this is furnished by observations on underground temperatures by Rambaut.[116] The curve of temperatures for March, 1899, shows a marked drop, or cold wave, of about 10 days' duration—whole period 20 days—the lowering (T_0) amounting to about 8.6°C. The magnitude of the temperature fall and lag of the minimum, as observed by platinum thermometers at various depths, is given in Table 5.2, and also the computed values. These latter were obtained by using the value of $\alpha = 0.0074$ cgs computed by Rambaut from the annual-wave curve. The computed temperature fall is of course half the range as determined from (5.3*a*).

More accurate calculations will be possible with the aid of the theory of Sec. 8.6.

TABLE 5.2

Depth, cm	Observed temperature fall, °C	Computed temperature fall, °C	Observed lag, days	Computed lag, days
0.0	8.6	8.6	0	0
16.5	5.9	6.7	1.4	0.8
45.7	3.4	4.2	2.5	2.3
107.9	1.3	1.6	4.9	5.4
174.0	0.33	0.57	8.0	8.7

5.11. Temperature Waves in Concrete. The above discussions may be applied at once to a mass of concrete as in a dam. Taking the diffusivity, *e.g.*, as 0.0058 cgs we may conclude that a cold wave of 3 days' duration (period 6 days), of minimum temperature $-20°C$ ($-4°F$), might cause the freezing temperature to penetrate a concrete mass at 4°C (39.2°F), a depth of some 56 cm (22 in.), while the annual variation of temperature at a depth of 2 m (6.6 ft) would be only 0.43 of what it is at the surface.

5.12. Periodic Flow and Climate; "Ice Mines." The annual periodic heat flow into the earth's surface in spring and summer and out in fall and winter tends to cause the seasons to lag behind the sun in phase and also may moderate slightly the annual temperature extremes. When we come to calculate this from (5.6*b*), however, we find that for soil ($k = 0.0022$, $\alpha = 0.0038$ cgs) it amounts to only about 1920 cal/cm^2 for the season, and for rock ($k = 0.006$, $\alpha = 0.010$ cgs), 3260 cal/cm^2, assuming an average annual surface temperature amplitude of 12°C. This would have its greatest effect in deep canyons where the large area of rock walls results in a marked reduction in the annual temperature range.

There are a number of well-known "ice mines" in the world. These are small regions, perhaps excavations, where the order of nature is reversed and ice forms in summer, while in winter the region is warmer than the surrounding locality. There seems to be no generally accepted explanation of this phenomenon, but it is undoubtedly connected with periodic heat inflow and outflow. It is doubtful if this explanation can be

sufficient in itself unless there is some way of increasing greatly the area of surface involved. This can happen if the local geologic structure, as in an immediately adjoining hill, is of a very porous character. In this case the whole hill might act like a gigantic calorimeter or regenerator, cooled by the winter winds to a considerable depth. This "cold" coming out in the form of cold air in summer could produce the freezing effects. It is suggested that this may be the explanation of the ice mine at the foot of a hill at Coudersport, Pa.*

5.13. Periodic Flow in Cylinder Walls. As another instance of periodic flow may be mentioned the heat penetration in the walls of a steam-engine cylinder. Callendar and Nicolson[24]† found that for 100 rpm the temperature range of the inner surface of the cylinder wall (cylinder head) during a cycle was 2.8°C (5.1°F). Using $\alpha = 0.121$ and $k = 0.108$ cgs, we find from (5.3*a*) that this variation is reduced at a depth of 0.25 cm (0.1 in.) to

$$2.8 \exp\left(-\frac{0.25}{\sqrt{0.121}}\sqrt{\frac{100\pi}{60}}\right) = 0.54°\text{C } (1°\text{F}) \qquad (a)$$

and at three times this depth to only 0.021°C (0.04°F). The heat flow into and out of the walls that takes place each cycle is given from (5.6*b*) as

$$\frac{Q}{A} = \sqrt{2}\,\frac{2.8 \times 0.108}{2 \times 0.348}\sqrt{\frac{60}{100\pi}} = 0.269 \text{ cal/cm}^2$$
$$= 0.99 \text{ Btu/ft}^2 \qquad (b)$$

This results in a loss of efficiency since it subtracts from the available energy during the power part of the stroke. To remedy this the "uniflow" engine is specially designed so that the steam enters at the ends and exhausts from the middle of the (long) cylinder. This involves smaller cyclical temperature changes of the cylinder walls and hence lessens the wasteful inflow and outflow of heat.

* See Lautensach[82] for an apparently similar case of cold-air storage but with a smaller temperature range.

† For a discussion of several of the other factors involved here see Janeway.[69] Also, see Meier.[96]

5.14. Thermal Stresses.* If a body or a portion of it is heated or cooled and at the same time constrained so that it cannot expand or contract, it will be subject to stress. Such stresses may be computed on the basis of the forces necessary to compress or extend the body from the dimensions it would take if allowed to expand or contract freely, back to its original ones.

If a bar of length L has its temperature raised from T_1 to T_2, it will, if allowed to expand freely, increase in length by an amount

$$\Delta L = \epsilon L(T_2 - T_1) \tag{a}$$

where ϵ is the coefficient of expansion. The stress, or force per unit area, necessary to compress the bar back to its original dimensions is

$$P = \frac{E\Delta L}{L} = E\epsilon(T_2 - T_1) \tag{b}$$

where E is the modulus of elasticity. This is then the stress required to keep it from expanding, in other words, the thermal stress in a constrained bar.

As an example, consider the stresses in tramway rails that have been welded together at a temperature of 40°F, if the rails are warmed to 95°F. If we take $E = 3 \times 10^7$ lb/in.^{2} and $\epsilon = 6.4 \times 10^{-6}$/°F for steel, we can compute at once from (b) that the stress would be 10,560 lb/in.^{2} compression. The customary burying of the body of the rail so that only its top surface is exposed affords some protection from the severity of daily temperature changes although very little for the annual, as the preceding theory readily shows.

In unconstrained bodies thermal stresses are produced by nonuniform temperature distribution. Examples of such occur in the warming up of steam turbine rotors and in the periodic heating and cooling of engine cylinder walls, or in the daily variation of surface temperatures in rocks, concrete structures, and the like. Such stresses may be taken as largely determined by the temperature gradient at the point. Differentiating

* See Timoshenko,[148, p. 203] Timoshenko and MacCullough,[149, p. 20] Kent,[75] and Roark.[118]

(5.2q), we have the expression, similar to (5.6a)

$$\frac{\partial T}{\partial x} = -T_0\sqrt{\frac{\pi}{\alpha P}}\,e^{-x\sqrt{\pi/\alpha P}}\left[\sin\left(\omega t - x\sqrt{\frac{\pi}{\alpha P}}\right) + \cos\left(\omega t - x\sqrt{\frac{\pi}{\alpha P}}\right)\right] \quad (c)$$

which shows that temperature stresses due to periodic variation are greatest for the surface layers of the material. It can be shown (Timoshenko[148, p. 212]) that for not too slow cyclical variations the stress is approximately given by the quantity $\epsilon E T_P/(1 - \nu)$, where T_P is the amplitude of the temperature variation at the point and ν is Poisson's ratio. For the cylinder wall of a diesel engine subject to surface-temperature fluctuations of ± 20°F we find, using the above value of ϵ and E for steel and putting $\nu = 0.3$, a stress of 5,500 lb/in.^{2} It is evident from (5.3a) that this would fall off rapidly below the surface, but that the rate of decrease would be less for a slow-running engine.

5.15. Problems

1. If the daily range of temperature at the surface of a soil of diffusivity 0.0049 cgs is 20°C, what is the range at 10 cm and 1 m below the surface?

Ans. 8.4°C; 0.0036°C

2. Solve the preceding problem for an annual range of 30°C and for depths of 10 cm, 1 m, and 10 m. *Ans.* 28.7°C; 19.1°C; 0.33°C

3. Compute the periodic heat flow into and out of the surface for the two preceding problems. (Use $k = 0.0037$ cgs.) *Ans.* 124 cal/cm^2; 3550 cal/cm^2

4. A long copper ($\alpha = 1.14$ cgs) rod is carefully insulated throughout its length and one end is alternately heated and cooled through the range 0 to 100°C every half-hour. Plot the temperatures along the bar for such time as will make the temperature of the heated end 50°C. Determine the wave length and velocity for this case; also, for the case in which the period is one-quarter hour.

Ans. $\lambda = 161$ cm, $V = 0.089$ cm/sec, for $P = \frac{1}{2}$ hr; $\lambda = 114$ cm, $V = 0.126$ cm/sec, for $P = \frac{1}{4}$ hr

5. A cold wave of 2 weeks' duration ($P = 4$ weeks) brings a temperature fall (amplitude) at the surface of 20°C. What will be the fall at a depth of 1 m in soil of diffusivity 0.0031 cgs and also of 0.0058 cgs? Also compute the time lag of the minimum in these cases.

Ans. 2.6°C, 4.5°C; 9.1, 6.7 days

CHAPTER 6

FOURIER SERIES

6.1. Before we can proceed further with our study of heat-conduction problems, we shall be obliged to take up the development of functions in trigonometric series. The necessity for this was apparent in Chap. 4 and could indeed be foreseen in the last chapter; for it was evident that, if the boundary condition had been expressed by other than a simple sine or cosine function, as it was, it could not have been satisfied by any of the solutions obtained, unless it should be of such a nature that it could be developed as a series of sine or cosine terms, in which case it might be possible to build up particular solutions to fit it.

Such a development was shown by Fourier to be possible for all functions that fulfill certain simple conditions. For example, the curve $y = f(x)$ may be represented between the limits $x = 0$ and $x = \pi$, by adding a series of sine curves, thus:

$$f(x) = y = a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \cdots \qquad (a)$$

or by a similar cosine series. The $f(x)$ can be represented in this way if it meets the following conditions within the range considered:

1. The $f(x)$ is single-valued: *i.e.*, for every value of x there is one and only one value of y (save at discontinuities).
2. The $f(x)$ is finite. For example, $f(x) = \tan x$ cannot be expanded in a Fourier series.
3. There are only a finite number of maxima and minima. For example, $f(x) = \sin 1/x$ cannot be so expanded.
4. The $f(x)$ is continuous, or at least has only a finite number of finite discontinuities.

The function that represents the initial state of temperature will satisfy these conditions, for there can be but a single value of the temperature at each point of a body, and this value must be finite. Furthermore, while there may exist initial discon-

tinuities, as at a surface of separation between two bodies, such discontinuities will always be finite. This indicates the applicability as well as the importance of Fourier's series in the theory of heat conduction.

6.2. Development in Sine Series. To accomplish this development it is necessary to find the values of the coefficients $a_1, a_2, a_3, \ldots$ of the series (6.1*a*). It is possible to find the value of a finite number, n, of these by solving n equations of the type

$$y_p = a_1 \sin x_p + a_2 \sin 2x_p + \cdots + a_n \sin nx_p \tag{a}$$

where x_p is one of n particular values of x chosen between 0 and π. This process also has the merit of making plausible the possibility of expanding a function in such a series; for with n terms the curve made up by summing the trigonometrical series coincides with the curve $y = f(x)$ at the n points and can be made identical with it if we take n large enough. But while this method is possible, it is not the simplest way, for the results may be obtained by a much shorter procedure, as follows:

We shall proceed on the assumption that the expansion (6.1*a*) is possible, and consider this assumption justified if we can find values for the coefficients. Multiply both sides of (6.1*a*) by $\sin mx\,dx$, where m is the number of the coefficient we wish to determine; then integrate from 0 to π:*

$$\begin{aligned}\int_0^\pi f(x) \sin mx\,dx = a_1 \int_0^\pi \sin mx \sin x\,dx &+ \cdots \\ + a_m \int_0^\pi \sin^2 mx\,dx &+ \cdots \\ + a_p \int_0^\pi \sin mx \sin px\,dx &+ \cdots \end{aligned} \tag{b}$$

Now $\displaystyle\int_0^\pi \sin mx \sin px\,dx = \frac{1}{2}\int_0^\pi \cos (p - m)x\,dx$

$$\begin{aligned}- \frac{1}{2}\int_0^\pi \cos (p + m)x\,dx = \frac{1}{2}\left[\frac{1}{p - m} \sin (p - m)x\right]_0^\pi \\ - \frac{1}{2}\left[\frac{1}{p + m} \sin (p + m)x\right]_0^\pi = 0\end{aligned} \tag{c}$$

* It can be shown that this procedure is essentially the same as that employed above if n is large. See Byerly.[23, p. 38]

Hence, the only term remaining on the right-hand side of (b) is

$$a_m \int_0^\pi \sin^2 mx\,dx = a_m \frac{\pi}{2} \tag{d}$$

Therefore,
$$a_m = \frac{2}{\pi} \int_0^\pi f(x) \sin mx\,dx \tag{e}$$

and the complete series may be written

$$f(x) = \frac{2}{\pi} \left\{ \left[\int_0^\pi f(x) \sin x\,dx \right] \sin x + \left[\int_0^\pi f(x) \sin 2x\,dx \right] \sin 2x + \cdots + \left[\int_0^\pi f(x) \sin nx\,dx \right] \sin nx + \cdots \right\} \tag{f}$$

6.3. As examples of the application of this series let us develop a few simple functions in this way.

(1) $f(x) = c$, any constant (Figs. 6.1a to d).

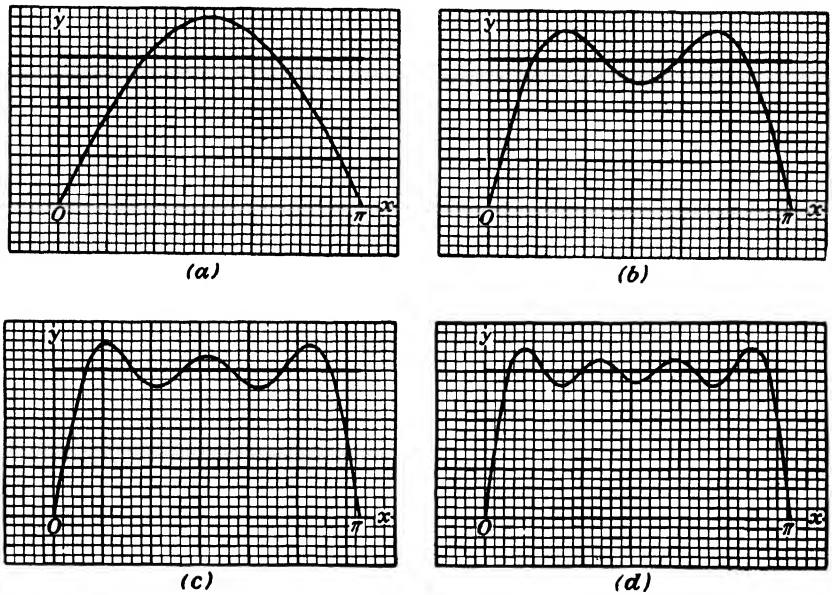

FIG. 6.1. The approximation curves for the sine series for $y = f(x)$, where $f(x) =$ a constant, $(0 < x < \pi)$. (a) One term, (b) two terms, (c) three terms, (d) four terms.

$$a_m = \frac{2}{\pi}\int_0^\pi c \sin mx\,dx = \frac{2c}{\pi}\int_0^\pi \sin mx\,dx \qquad (a)$$

$$= \frac{2c}{\pi m}[1 - (-1)^m] \qquad (b)$$

$$= 0 \text{ if } m \text{ is even} \qquad (c)$$

$$= \frac{4c}{\pi m} \text{ if } m \text{ is odd} \qquad (d)$$

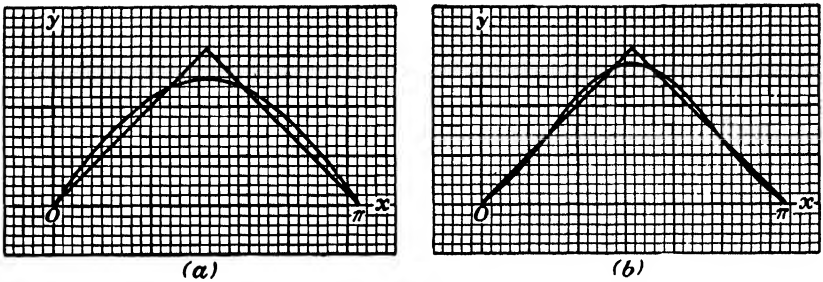

FIG. 6.2. The approximation curves for the sine series for $y = f(x)$, where $f(x) = x$, $(0 < x < \pi/2)$; $f(x) = \pi - x$, $(\pi/2 < x < \pi)$. (a) One term, (b) two terms.

Hence, the even terms will be lacking, and we get

$$f(x) \equiv c = \frac{4c}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right) \qquad (e)$$

For $x = \pi/2$, this enables us to write the expansion for $\pi/4$ thus:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

(2) Let us reproduce the curve (Figs. 6.2*a* and *b*)

$$f(x) = x \text{ from } x = 0 \text{ to } x = \frac{\pi}{2} \qquad (f)$$

$$f(x) = \pi - x \text{ from } x = \frac{\pi}{2} \text{ to } x = \pi \qquad (g)$$

$$a_m = \frac{2}{\pi}\int_0^{\pi/2} f(x)\sin mx\,dx + \frac{2}{\pi}\int_{\pi/2}^{\pi} f(x)\sin mx\,dx \qquad (h)$$

$$= \frac{2}{\pi}\int_0^{\pi/2} x\sin mx\,dx + \frac{2}{\pi}\int_{\pi/2}^{\pi} (\pi - x)\sin mx\,dx \qquad (i)$$

$$= \frac{2}{\pi}\left[\left(\frac{\sin mx}{m^2} - \frac{x \cos mx}{m}\right)_0^{\pi/2} + \pi\left(-\frac{1}{m}\cos mx\right)_{\pi/2}^{\pi} - \left(\frac{\sin mx}{m^2} - \frac{x \cos mx}{m}\right)_{\pi/2}^{\pi}\right] \qquad (j)$$

$$= \frac{4}{m^2\pi}\sin m\frac{\pi}{2} \qquad (k)$$

$$\left.\begin{array}{llrl} \text{If } m = 1, \text{ or } 4p+1, & \sin m\frac{\pi}{2} & = & 1 \\ m = 2, \text{ or } 4p+2, & \sin m\frac{\pi}{2} & = & 0 \\ m = 3, \text{ or } 4p+3, & \sin m\frac{\pi}{2} & = & -1 \\ m = 4, \text{ or } 4p+4, \text{ or } 4p, & \sin m\frac{\pi}{2} & = & 0 \\ & \multicolumn{3}{l}{\text{where } p \text{ is any integer}} \end{array}\right\} \qquad (l)$$

Again, the even terms are absent, and

$$f(x) = \frac{4}{\pi}\left(\frac{\sin x}{1} - \frac{\sin 3x}{9} + \frac{\sin 5x}{25} - \cdots\right) \qquad (m)$$

For $x = \pi/2$ this gives

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots \qquad (n)$$

(3) Finite discontinuity (Figs. 6.3*a* to *f*).

$$f(x) = x \text{ from } x = 0 \text{ to } x = \frac{\pi}{2} \qquad (o)$$

$$= 0 \text{ from } x = \frac{\pi}{2} \text{ to } x = \pi \qquad (p)$$

Breaking up a_m into two parts and substituting the values for $f(x)$, we get

$$a_m = \frac{2}{\pi}\int_0^{\pi/2} x \sin mx\,dx + \frac{2}{\pi}\int_{\pi/2}^{\pi} 0 \sin mx\,dx = \frac{2}{\pi}\int_0^{\pi/2} x \sin mx\,dx \qquad (q)$$

$$= \frac{2}{\pi}\left(\frac{\sin mx}{m^2} - \frac{x \cos mx}{m}\right)_0^{\pi/2} \qquad (r)$$

$$
\left.\begin{aligned}
&= \frac{2}{\pi}\left(\frac{1}{m^2}\right) && \text{if } m = 4p + 1 \\
&= \frac{2}{\pi}\left(\frac{m\pi}{2m^2}\right) && \text{if } m = 4p + 2 \\
&= \frac{2}{\pi}\left(-\frac{1}{m^2}\right) && \text{if } m = 4p + 3 \\
&= \frac{2}{\pi}\left(-\frac{m\pi}{2m^2}\right) && \text{if } m = 4p + 4
\end{aligned}\right\} \tag{s}
$$

$$
\therefore f(x) = \frac{2}{\pi}\left(\frac{\sin x}{1} + \frac{\pi \sin 2x}{4} - \frac{\sin 3x}{9} - \frac{2\pi \sin 4x}{16} + \frac{\sin 5x}{25} + \frac{3\pi \sin 6x}{36} - \cdots\right) \tag{t}
$$

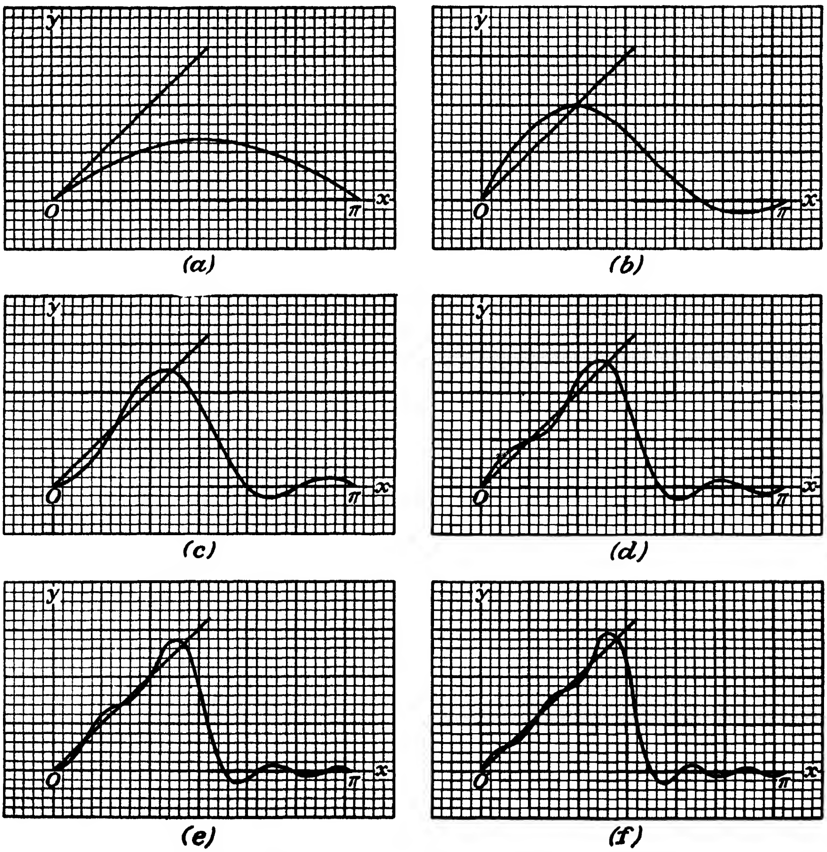

FIG. 6.3. The approximation curves for the sine series for $y = f(x)$, where $f(x) = x$, $(0 < x < \pi/2)$; $f(x) = 0$, $(\pi/2 < x < \pi)$. (a) One term, (b) two

It may be noted that at the point of discontinuity,* $x = \pi/2$, the value of the series is

$$\frac{2}{\pi}\left(\frac{1}{1} + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots\right) = \frac{2}{\pi}\left(\frac{\pi^2}{8}\right) = \frac{\pi}{4} \qquad (u)$$

which is the mean of the values approached by the function as x approaches $\pi/2$ from opposite sides.

6.4. Development in Cosine Series. In a manner quite similar to the foregoing we are also able to develop such functions as fulfill the conditions we have mentioned, in cosine series between the limits $x = 0$ and $x = \pi$. Thus,

$$f(x) = b_0' + b_1 \cos x + b_2 \cos 2x + b_3 \cos 3x + \cdots \qquad (a)$$

The constant term that appears here, though not in the sine series, may be thought of as the coefficient of a term $b_0' \cos (0\,x)$, which shows at once why the corresponding term for the sine series is lacking.

To find the value of any coefficient b_m, we proceed as before, multiplying both sides of (a) by $\cos mx\,dx$ and integrating from 0 to π; then, since terms of the type

$$\int_0^\pi b_p \cos px \cos mx\,dx \qquad (b)$$

vanish just as did similar terms in (6.2c), we have remaining on the right-hand side only

$$b_m \int_0^\pi \cos^2 mx\,dx = \frac{b_m}{2m}\left[(mx + \cos mx \sin mx)\right]_0^\pi \qquad (c)$$

$$= \frac{\pi}{2} b_m \qquad \text{if } m \neq 0$$

$$\therefore\ b_m = \frac{2}{\pi}\int_0^\pi f(x) \cos mx\,dx \qquad (d)$$

To get b_0' we must multiply (a) by dx only and integrate from 0 to π; then,

$$\int_0^\pi f(x)\,dx = \int_0^\pi b_0'\,dx + \int_0^\pi b_1 \cos x\,dx + \cdots = b_0'\pi \qquad (e)$$

* It is seen that the representation of the curve (see Figs. 6.3f and 6.1d) is not as perfect near the discontinuities as elsewhere. This is known as the "Gibbs' phenomenon." See Carslaw,[26] Churchill.[32, p.86]

since all terms but the first vanish. Therefore,

$$b'_0 = \frac{1}{\pi}\int_0^\pi f(x)\,dx \qquad (f)$$

This is just half the value that (d) would give if $m = 0$ were substituted; therefore, to save an extra formula, (a) is generally written

$$f(x) = \tfrac{1}{2}b_0 + b_1 \cos x + b_2 \cos 2x + b_3 \cos 3x + \cdots \qquad (g)$$

where the value of any coefficient, including the first, is given by (d). The complete cosine series may then be written

$$\begin{aligned} f(x) = \frac{2}{\pi}\Bigg\{\frac{1}{2}\int_0^\pi f(x)\,dx &+ \left[\int_0^\pi f(x)\cos x\,dx\right]\cos x \\ &+ \left[\int_0^\pi f(x)\cos 2x\,dx\right]\cos 2x + \cdots \\ &+ \left[\int_0^\pi f(x)\cos mx\,dx\right]\cos mx + \cdots\Bigg\} \qquad (h)\end{aligned}$$

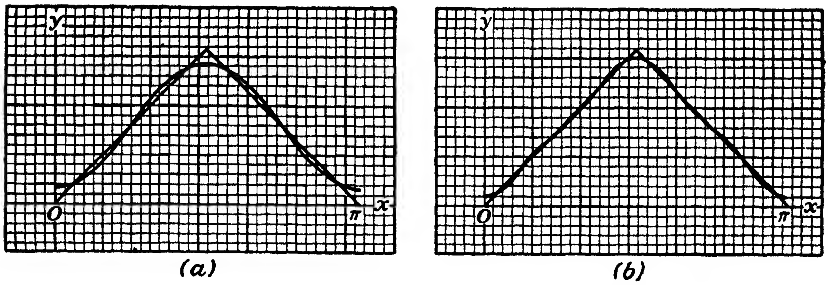

FIG. 6.4. The approximation curves for the cosine series for $y = f(x)$, where $f(x) = x$, $(0 < x < \pi/2)$; $f(x) = \pi - x$, $(\pi/2 < x < \pi)$. (a) Constant term and next term, (b) constant term and next two terms.

6.5. As an example take the same function as we developed in a sine series under (2) in Sec. 6.3 (see Figs. 6.2a and b and 6.4a and b):

$$f(x) = x \text{ from } x = 0 \text{ to } x = \frac{\pi}{2}$$

$$f(x) = \pi - x \text{ from } x = \frac{\pi}{2} \text{ to } x = \pi$$

Then, $$b_m = \frac{2}{\pi}\left[\int_0^{\pi/2} x\cos mx\,dx + \int_{\pi/2}^{\pi}(\pi - x)\cos mx\,dx\right] \qquad (a)$$

$$= \frac{2}{\pi}\left(\frac{\cos mx + mx \sin mx}{m^2}\right)_0^{\pi/2} + \frac{2}{\pi}\frac{\pi}{m}\left(\sin mx\right)_{\pi/2}^{\pi}$$

$$- \frac{2}{\pi}\left(\frac{\cos mx + mx \sin mx}{m^2}\right)_{\pi/2}^{\pi} \quad \text{when } m \neq 0 \quad (b)$$

$$= \frac{2}{\pi}\left(\frac{\cos m\pi/2}{m^2} + \frac{\pi}{2m}\sin\frac{m\pi}{2} - \frac{1}{m^2} - \frac{\pi}{m}\sin\frac{m\pi}{2}\right.$$

$$\left. - \frac{\cos m\pi}{m^2} + \frac{\cos m\pi/2}{m^2} + \frac{\pi}{2m}\sin\frac{m\pi}{2}\right) \quad (c)$$

$$= \frac{2}{\pi m^2}\left(2\cos\frac{m\pi}{2} - \cos m\pi - 1\right) \quad (d)$$

$$\left.\begin{aligned}
&\text{If } m = 1 \text{ or } 4p+1, \text{ bracket} = 0 \\
&\qquad\qquad \therefore b_m = 0 \\
&m = 2 \text{ or } 4p+2, \text{ bracket} = -4 \\
&\qquad\qquad \therefore b_m = -\frac{2}{\pi}\frac{1}{(m/2)^2} \\
&m = 3 \text{ or } 4p+3, \text{ bracket} = 0 \\
&\qquad\qquad \therefore b_m = 0 \\
&m = 4 \text{ or } 4p+4, \text{ bracket} = 0 \\
&\qquad\qquad \therefore b_m = 0
\end{aligned}\right\} \quad (e)$$

To get b_0, substitute $m = 0$ in (a) and integrate Then,

$$b_0 = \frac{2}{\pi}\int_0^{\pi/2} x\,dx + \frac{2}{\pi}\left(\pi\int_{\pi/2}^{\pi} dx - \int_{\pi/2}^{\pi} x\,dx\right) \quad (f)$$

$$= \frac{\pi}{4} + \pi - \frac{3\pi^2}{4\pi} = \frac{\pi}{2} \quad (g)$$

$$\therefore \frac{1}{2}b_0 = \frac{\pi}{4} \quad (h)$$

So, finally, we have

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left(\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + \cdots\right) \quad (i)$$

to represent the same curve as is given by the sine series (6.3m).

6.6. The Complete Fourier Series. It is possible to combine the sine series and the cosine series so as to expand any function satisfying our original conditions (Sec. 6.1) between $-\pi$ and π. This gives the true or complete Fourier series

$$f(x) = \tfrac{1}{2}b_0 + b_1 \cos x + b_2 \cos 2x + \cdots + a_1 \sin x + a_2 \sin 2x + \cdots \quad (a)$$

The coefficients $a_1, a_2 \ldots b_0, b_1, b_2, \ldots$ may be determined

in much the same way as before. Multiply both sides of (a) by $\sin mx\,dx$ and integrate from $-\pi$ to π. Then,

$$\int_{-\pi}^{\pi} f(x) \sin mx\,dx = \frac{1}{2} b_0 \int_{-\pi}^{\pi} \sin mx\,dx$$
$$+ b_1 \int_{-\pi}^{\pi} \sin mx \cos x\,dx + \cdots + b_p \int_{-\pi}^{\pi} \sin mx \cos px\,dx + \cdots$$
$$+ a_1 \int_{-\pi}^{\pi} \sin mx \sin x\,dx + \cdots + a_m \int_{-\pi}^{\pi} \sin^2 mx\,dx + \cdots$$
$$+ a_p \int_{-\pi}^{\pi} \sin mx \sin px\,dx + \cdots \qquad (b)$$

Now $\displaystyle\int_{-\pi}^{\pi} \sin mx\,dx = 0$ and $\displaystyle\int_{-\pi}^{\pi} \sin mx \cos mx\,dx = 0$ $\qquad (c)$

Also (see Appendix B)

$$\int_{-\pi}^{\pi} \sin mx \cos px\,dx = \left[-\frac{\cos (m-p)x}{2(m-p)} - \frac{\cos (m+p)x}{2(m+p)}\right]_{-\pi}^{\pi} = 0 \qquad (d)$$

and

$$\int_{-\pi}^{\pi} \sin mx \sin px\,dx = \left[\frac{\sin (m-p)x}{2(m-p)} - \frac{\sin (m+p)x}{2(m+p)}\right]_{-\pi}^{\pi} = 0 \qquad (e)$$

Hence, the only term remaining on the right-hand side of (b) is

$$a_m \int_{-\pi}^{\pi} \sin^2 mx\,dx = a_m\pi \qquad (f)$$

Therefore, $$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin mx\,dx \qquad (g)$$

In the same fashion we can, after multiplication of (a) by $\cos mx\,dx$ and integrating, determine

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos mx\,dx \qquad (h)$$

which also holds for $m = 0$.

Since x will generally refer in our conduction problems to some particular point or plane in a body, it is better to use some variable of integration such as λ in writing (g) and (h), which then become

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\lambda) \sin m\lambda \, d\lambda \qquad (i)$$

and

$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\lambda) \cos m\lambda \, d\lambda \qquad (j)$$

6.7. It is instructive to get expressions (6.6i,j) by another method. We have seen that any function of the kind considered can be represented by either a sine or cosine development for all values of x between 0 and π. We may now question what such series would give at and beyond these limits. Obviously, the sine series can hold at the limits $x = 0$ and $x = \pi$ only when the $f(x)$ is itself zero at these points, although it will

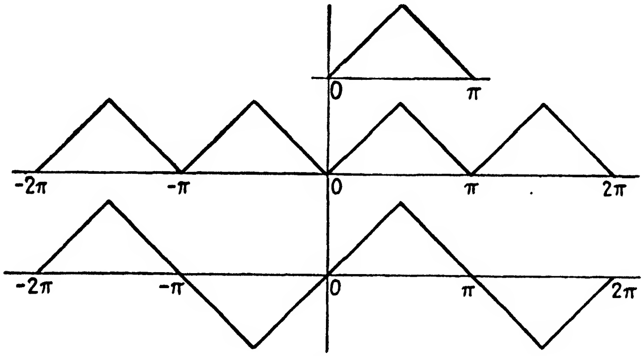

FIG. 6.5. Curves showing the results of extending the limits beyond 0 and π. The cosine development for (a) gives a curve like (b), while the sine series for (a) gives (c).

hold for points infinitesimally near these limits for any value of $f(x)$. For example, it breaks down at the limits in the case of $f(x) = c$ already given.

Both series are periodic and afford curves that must repeat themselves whenever x is changed by 2π; and, as both series give the same curve between 0 and π, the difference, if any, between the curves given by the two series must come between π and 2π, or, what amounts to the same thing, between 0 and $-\pi$. This difference is at once evident if we consider that the values of the sine terms will change sign with change to negative angle, while the cosine terms will not. Thus, the sine and cosine developments, when extended beyond the limits 0 and π, give curves of the type shown in Fig. 6.5. We may conclude from this,

then, that if $f(x)$ is an *even* function, *i.e.*, if $f(x) = f(-x)$, it may be represented by a cosine series from $-\pi$ to π. Similarly, an *odd* function $[f(x) = -f(-x)]$ will be given by a sine series for these same limits. Not all functions are either odd or even, *e.g.*, e^x, but it is possible to express any function as the sum of an odd and an even function; thus,

$$f(x) \equiv \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} \tag{a}$$

the first term being even, since it does not change sign with x, while the second does and is therefore odd. To expand any function satisfying our primitive conditions, then, between $x = -\pi$ and π, we may write (6.6*a*) where the coefficients are determined from (6.2*e*) and (6.4*d*) as

$$a_m = \frac{2}{\pi}\int_0^\pi \frac{f(x) - f(-x)}{2} \sin mx\, dx \tag{b}$$

and

$$b_m = \frac{2}{\pi}\int_0^\pi \frac{f(x) + f(-x)}{2} \cos mx\, dx \tag{c}$$

Since the values of definite integrals are functions only of the limits and not of the variable of integration, we may replace x in these expressions by any other variable λ; thus,

$$a_m = \frac{2}{\pi}\int_0^\pi \frac{f(\lambda) - f(-\lambda)}{2} \sin m\lambda\, d\lambda \tag{d}$$

and

$$b_m = \frac{2}{\pi}\int_0^\pi \frac{f(\lambda) + f(-\lambda)}{2} \cos m\lambda\, d\lambda \tag{e}$$

We can simplify expressions (*d*) and (*e*) somewhat, for the former is equivalent to

$$\frac{1}{\pi}\left[\int_0^\pi f(\lambda) \sin m\lambda\, d\lambda - \int_0^\pi f(-\lambda) \sin m\lambda\, d\lambda\right] \tag{f}$$

and if we replace λ by $-\lambda'$ in the second integral, it is transformed into

$$-\int_0^{-\pi} f(\lambda') \sin m\lambda'\, d\lambda' \tag{g}$$

This is equal to

$$+\int_{-\pi}^0 f(\lambda') \sin m\lambda'\, d\lambda' \tag{h}$$

which, since it is immaterial what symbol is used for the integration variable, may as well be written

$$+\int_{-\pi}^{0} f(\lambda) \sin m\lambda \, d\lambda \qquad (i)$$

Hence, we have $$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\lambda) \sin m\lambda \, d\lambda \qquad (j)$$

In a similar way we obtain

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\lambda) \cos m\lambda \, d\lambda \qquad (k)$$

6.8. Change of the Limits. While our expansion as heretofore considered holds only for the region $x = -\pi$ to $x = \pi$, we can, by a simple change of variable, make it hold from $x = -l$ to l. For let

$$z = \frac{\pi x}{l}; \qquad \text{then } f(x) = f\left(\frac{lz}{\pi}\right) = F(z)$$

$$\therefore f(x) = F(z) = \tfrac{1}{2}b_0 + b_1 \cos z + b_2 \cos 2z + \cdots$$
$$+ a_1 \sin z + a_2 \sin 2z + \cdots \qquad (a)$$

for values of z from $-\pi$ to π, and

$$f(x) = \frac{1}{2} b_0 + b_1 \cos\frac{\pi x}{l} + b_2 \cos\frac{2\pi x}{l} + \cdots$$
$$+ a_1 \sin\frac{\pi x}{l} + a_2 \sin\frac{2\pi x}{l} + \cdots \qquad (b)$$

for values of x from $-l$ to l, where

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} F(z) \cos mz \, dz = \frac{1}{l}\int_{-l}^{l} f(x) \cos\frac{m\pi x}{l}\, dx \qquad (c)$$

since $z = \pi x/l$, and $dz = \pi\, dx/l$. This may also be written

$$b_m = \frac{1}{l}\int_{-l}^{l} f(\lambda) \cos\frac{m\pi\lambda}{l}\, d\lambda \qquad (d)$$

Similarly, $$a_m = \frac{1}{l}\int_{-l}^{l} f(\lambda) \sin\frac{m\pi\lambda}{l}\, d\lambda \qquad (e)$$

In the same way the sine series (6.1a) may be written

$$f(x) = a_1 \sin \frac{\pi x}{l} + a_2 \sin \frac{2\pi x}{l} + \cdots \tag{f}$$

where
$$a_m = \frac{2}{l} \int_0^l f(\lambda) \sin \frac{m\pi\lambda}{l}\, d\lambda \tag{g}$$

while (6.4g) becomes

$$f(x) = \frac{1}{2} b_0 + b_1 \cos \frac{\pi x}{l} + b_2 \cos \frac{2\pi x}{l} + \cdots \tag{h}$$

where
$$b_m = \frac{2}{l} \int_0^l f(\lambda) \cos \frac{m\pi\lambda}{l}\, d\lambda \tag{i}$$

While series (b) applies generally, (f) and (h) hold only from $x = 0$ to l, unless $f(x)$ is an even function, in which case the cosine series will be good from $-l$ to l, while if odd, the sine series will hold over this range.

6.9. Fourier's Integral. In the foregoing we have developed $f(x)$ into a Fourier's series that represented the function from $-l$ to l where l may have any value whatever. We shall now proceed to express the sum of such a series in the form of an integral and, by allowing the limits to extend indefinitely, obtain an expression that holds for all values of x. Write the series (6.8b) with the aid of (6.8d) and (6.8e).

$$\begin{aligned} f(x) = \frac{1}{l} \bigg[\frac{1}{2} \int_{-l}^{l} f(\lambda)\, d\lambda &+ \int_{-l}^{l} f(\lambda) \cos \frac{\pi\lambda}{l} \cos \frac{\pi x}{l}\, d\lambda \\ &+ \int_{-l}^{l} f(\lambda) \cos \frac{2\pi\lambda}{l} \cos \frac{2\pi x}{l}\, d\lambda + \cdots \\ &+ \int_{-l}^{l} f(\lambda) \sin \frac{\pi\lambda}{l} \sin \frac{\pi x}{l}\, d\lambda \\ &+ \int_{-l}^{l} f(\lambda) \sin \frac{2\pi\lambda}{l} \sin \frac{2\pi x}{l}\, d\lambda + \cdots \bigg] \end{aligned} \tag{a}$$

When terms are collected, this becomes

$$\begin{aligned} f(x) = \frac{1}{l} \int_{-l}^{l} f(\lambda)\, d\lambda \bigg(\frac{1}{2} &+ \sum_{m=1}^{\infty} \cos \frac{m\pi\lambda}{l} \cos \frac{m\pi x}{l} \\ &+ \sum_{m=1}^{\infty} \sin \frac{m\pi\lambda}{l} \sin \frac{m\pi x}{l} \bigg) \end{aligned} \tag{b}$$

But since $\cos r \cos s + \sin r \sin s = \cos (r - s)$, this may be written

$$f(x) = \frac{1}{l}\int_{-l}^{l} f(\lambda)\,d\lambda\left[\frac{1}{2} + \sum_{m=1}^{\infty} \cos\frac{m\pi}{l}(\lambda - x)\right] \qquad (c)$$

or, if we remember that $\cos (\varphi) = \cos (-\varphi)$,

$$f(x) = \frac{1}{2l}\int_{-l}^{l} f(\lambda)\,d\lambda\left[1 + \sum_{m=1}^{\infty} \cos\frac{m\pi}{l}(\lambda - x) + \sum_{m=-1}^{-\infty} \cos\frac{m\pi}{l}(\lambda - x)\right] \qquad (d)$$

$$= \frac{1}{2\pi}\int_{-l}^{l} f(\lambda)\,d\lambda\left[\frac{\pi}{l}\sum_{m=-\infty}^{\infty} \cos\frac{m\pi}{l}(\lambda - x)\right] \qquad (e)$$

since $\cos (0\,\pi/l)(\lambda - x) = 1$. As l increases indefinitely, we may write $\gamma \equiv m\pi/l$ and $d\gamma = \pi/l$, and the expression in braces in (e) then becomes

$$\int_{-\infty}^{\infty} \cos \gamma(\lambda - x)\,d\gamma \qquad (f)$$

Therefore, $$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\lambda)\,d\lambda \int_{-\infty}^{\infty} \cos \gamma(\lambda - x)\,d\gamma \qquad (g)$$

an expression holding for all values of x and for the same class of functions as previously defined. It is known as "Fourier's integral."

6.10. Equation (6.9g) can be given a slightly different form by means of the following deduction, which will prove of use: For any function, $\varphi(\lambda)$,

$$\int_{-l}^{l} \varphi(\lambda)\,d\lambda = \int_{0}^{l} \varphi(\lambda)\,d\lambda + \int_{-l}^{0} \varphi(\lambda)\,d\lambda \qquad (a)$$

In the last term substitute $-\lambda'$ for λ; then,

$$\int_{-l}^{0} \varphi(\lambda)\,d\lambda = -\int_{l}^{0} \varphi(-\lambda')\,d\lambda' \qquad (b)$$

$$= -\int_{l}^{0} \varphi(-\lambda)\,d\lambda \qquad (c)$$

since its value is independent of the integration variable [see (6.7*i*)]. If $\varphi(\lambda)$ is even, *i.e.*, if $\varphi(\lambda) = \varphi(-\lambda)$, (*c*) means that

$$\int_{-l}^{0} \varphi(\lambda)\,d\lambda = -\int_{l}^{0} \varphi(\lambda)\,d\lambda = \int_{0}^{l} \varphi(\lambda)\,d\lambda \tag{d}$$

so that

$$\int_{-l}^{l} \varphi(\lambda)\,d\lambda = 2\int_{0}^{l} \varphi(\lambda)\,d\lambda \tag{e}$$

while if $\varphi(\lambda)$ is odd,

$$\int_{-l}^{l} \varphi(\lambda)\,d\lambda = \int_{0}^{l} \varphi(\lambda)\,d\lambda - \int_{0}^{l} \varphi(\lambda)\,d\lambda = 0 \tag{f}$$

Since the cosine is an even function, we may write at once, instead of (6.9*g*),

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\lambda)\,d\lambda \int_{0}^{\infty} \cos\gamma(\lambda - x)\,d\gamma \tag{g}$$

6.11. Again, if $f(x)$ is either odd or even, we may put (6.10*g*) in somewhat simpler form. Since the limits of integration in (6.10*g*) do not contain either λ or γ, the integration may be performed in either of two possible orders; *i.e.*,

$$\int_{-\infty}^{\infty} f(\lambda)\,d\lambda \int_{0}^{\infty} \cos\gamma(\lambda - x)\,d\gamma = \int_{0}^{\infty} d\gamma \int_{-\infty}^{\infty} f(\lambda)\cos\gamma(\lambda - x)\,d\lambda \tag{a}$$

Now

$$\int_{-\infty}^{\infty} f(\lambda)\cos\gamma(\lambda - x)\,d\lambda = \int_{0}^{\infty} f(\lambda)\cos\gamma(\lambda - x)\,d\lambda + \int_{-\infty}^{0} f(\lambda)\cos\gamma(\lambda - x)\,d\lambda \tag{b}$$

and, following the general methods of the previous section, we may write the last term

$$\int_{-\infty}^{0} f(\lambda)\cos\gamma(\lambda - x)\,d\lambda$$

$$= -\int_{\infty}^{0} f(-\lambda')\cos\gamma(-\lambda' - x)\,d\lambda' \tag{c}$$

$$= \int_{0}^{\infty} f(-\lambda')\cos\gamma(\lambda' + x)\,d\lambda' \tag{d}$$

$$= \int_{0}^{\infty} f(-\lambda)\cos\gamma(\lambda + x)\,d\lambda \tag{e}$$

$$= -\int_0^\infty f(\lambda) \cos \gamma(\lambda + x)\, d\lambda \quad \text{if } f(\lambda) \text{ is odd} \qquad (f)$$

$$= \int_0^\infty f(\lambda) \cos \gamma(\lambda + x)\, d\lambda \quad \text{if } f(\lambda) \text{ is even} \qquad (g)$$

Therefore, if $f(x)$ is odd, (6.10g) becomes, for all values of x,

$$f(x) = \frac{1}{\pi}\int_0^\infty d\gamma \int_0^\infty f(\lambda)[\cos \gamma(\lambda - x) - \cos \gamma(\lambda + x)]\, d\lambda \qquad (h)$$

$$= \frac{2}{\pi}\int_0^\infty d\lambda \int_0^\infty f(\lambda) \sin \gamma\lambda \sin \gamma x\, d\gamma \qquad (i)$$

while, if it is even, we have, instead,

$$f(x) = \frac{1}{\pi}\int_0^\infty d\gamma \int_0^\infty f(\lambda)[\cos \gamma(\lambda - x) + \cos \gamma(\lambda + x)]\, d\lambda \qquad (j)$$

$$= \frac{2}{\pi}\int_0^\infty d\lambda \int_0^\infty f(\lambda) \cos \gamma\lambda \cos \gamma x\, d\gamma \qquad (k)$$

Equations (i) and (k) hold for all positive values of x in the case of any function.

6.12. Harmonic Analyzers. The analytical development of a function in a Fourier's series, with the determination of a large number of coefficients, is well-nigh impossible in many cases, and in any event involves considerable computation. To eliminate this there have been invented several machines that are designed to compound automatically a limited number of sine or cosine terms into the resulting curve, or to perform the more difficult inverse process of analyzing a given function into its component Fourier's series. One of the earliest of these has become well known because of its great simplicity, as well as from the fame of its designer, Lord Kelvin.* A long cord or tape is passed over a series of fixed and movable pulleys, to each of which a simple harmonic motion of appropriate period and amplitude is given. The end of the cord will then have a displacement at each instant equal to double the sum of the displacements of the movable pulleys. This principle has been extensively developed† in machines of 40 or more elements, and Michelson and Stratton[97] have devised a machine of 80 elements using a

* See Thomson and Tait.[147, I, p. 44]

† See Kranz[79] and Miller.[98]

spring arrangement instead of the cord. Various electrical methods have also been developed.

In such a machine of 40 elements the frequencies of the elements are 1,2,3 . . . 40 times that of the fundamental. The process of combining sine or cosine terms is that of giving each element an amplitude of the proper magnitude and sign. The sum of all the terms appears in the displacement of a pen draw-

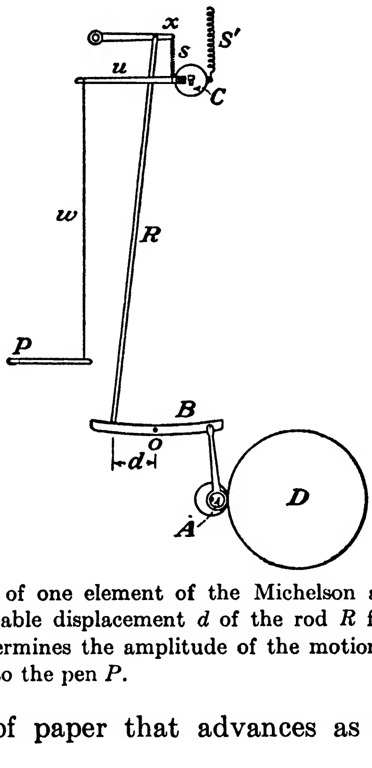

FIG. 6.6. Section of one element of the Michelson and Stratton harmonic analyzer. The adjustable displacement d of the rod R from the center of the oscillating arm B determines the amplitude of the motion. The sum of all the effects is transmitted to the pen P.

ing on a sheet of paper that advances as the instrument is operated.

One such element, for the Michelson and Stratton analyzer, is shown in Fig. 6.6. The wheel D is of such size as to give the eccentric A the proper frequency, and the desired amplitude is secured by adjusting the rod R on the lever B. The corresponding harmonic stretching of the spring s causes, along with that of all the other elements, a pull on the cylinder C. This gives a vertical motion to the pen P, that writes on paper carried on a plate moving horizontally as the machine is operated so as

to trace a curve that represents the sum of the contributions of all the elements.

6.13. The method of reversing this process and finding for any given function the coefficients of the corresponding Fourier's series may be seen from the following considerations:

Suppose we wish to develop a function in terms of the sine series. Then,

$$f(x) = a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \cdots \qquad (a)$$

where
$$a_p = \frac{2}{\pi}\int_0^\pi f(x) \sin px\,dx \qquad (b)$$

$$= \frac{2dx}{\pi}[f(x_1) \sin px_1 + f(x_2) \sin px_2 + \cdots + f(x_{40}) \sin px_{40}] \qquad (c)$$

if we replace the integral by a series and consider that we have a 40-element machine. Now, let $x_2 = 2x_1$, $x_3 = 3x_1$, . . . $x_{40} = 40x_1$. Then, (*c*) becomes

$$a_p = \frac{2dx}{\pi}[f(x_1) \sin px_1 + f(2x_1) \sin 2px_1 + \cdots + f(40x_1) \sin 40px_1] \qquad (d)$$

To analyze a curve divide it into 40 equal parts whose abscissas have the values $\pi/40$, $2\pi/40$, . . . π and adjust the amplitudes of the 40 elements of the machine proportionally to the 40 ordinates of these parts. As the analyzer is operated, the slowest turning or fundamental element will, at any instant, have turned through an angle px_1, and the paper will have advanced a distance proportional to p, say, p cm. We see then from (*d*) that for $p = 1$ the coefficient a_1 is given by the ordinate of the curve drawn by the analyzer at a distance of 1 cm (*i.e.*, $p = 1$) from the origin. Similarly, a_2 is the ordinate at 2 cm, and the other ordinates are obtained in the same fashion. When the curve has been completed, it is evident that the slowest element has rotated π radians and the fastest 40π.

Such instruments are of great usefulness in analyzing sound waves, alternating-current waves, and various other curves.*

* For a simple graphical method of analysis see Slichter.[132]

6.14. Problems

1. Develop the sine series that gives $y = 0$ for x between 0 and $\pi/2$; and $y = c$ for x between $\pi/2$ and π. Plot and add the first four or five terms.

$$Ans.\ y = \frac{2c}{\pi}\left(\frac{\sin x}{1} - \frac{2\sin 2x}{2} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} - \frac{2\sin 6x}{6} + \cdots\right)$$

2. Do this for the corresponding cosine series.

$$Ans.\ y = \frac{2c}{\pi}\left(\frac{\pi}{4} - \frac{\cos x}{1} + \frac{\cos 3x}{3} - \frac{\cos 5x}{5} + \cdots\right)$$

3. Show that $x = 2\left(\dfrac{\sin x}{1} - \dfrac{\sin 2x}{2} + \dfrac{\sin 3x}{3} - \cdots\right)$ for x between 0 and π.

4. Develop $f(x)$ in a sine series if $f(x) = c/3$ for $x = 0$ to $l/3$; $f(x) = 0$, for $x = l/3$ to $2l/3$; $f(x) = -c/3$, for $x = 2l/3$ to l.

$$Ans.\ f(x) = \frac{c}{\pi}\left(\sin\frac{2\pi x}{l} + \frac{1}{2}\sin\frac{4\pi x}{l} + \frac{1}{4}\sin\frac{8\pi x}{l} + \frac{1}{5}\sin\frac{10\pi x}{l} + \cdots\right)$$

5. Verify

$$e^x = 2l\left(\frac{1}{2}\frac{e^l - 1}{l^2} - \frac{e^l + 1}{l^2 + \pi^2}\cos\frac{\pi x}{l} + \frac{e^l - 1}{l^2 + 4\pi^2}\cos\frac{2\pi x}{l} - \frac{e^l + 1}{l^2 + 9\pi^2}\cos\frac{3\pi x}{l} + \cdots\right) \text{ from } x = 0 \text{ to } x = l$$

6. If $f(x) = 0$ from $x = -\pi$ to 0; and $f(x) = x$ from $x = 0$ to π, show that

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right) + \left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right)$$

7. Develop $c + \sin x$ in a cosine series between 0 and π; and in a complete Fourier's series between $-\pi$ and π.

$$Ans.\ y = c + \frac{2}{\pi}\left(1 - \frac{2}{3}\cos 2x - \frac{2}{15}\cos 4x - \frac{2}{35}\cos 6x + \cdots\right);$$

$$y = c + \sin x$$

8. Outline the curve between $-\pi$ and π, formed by the addition of series (6.3m) and (6.5i).

9. With the aid of (6.7a) graph the two functions, even and odd, whose sum is the curve $f(x) = x$ for x positive and $f(x) = c$ for x negative.

CHAPTER 7

LINEAR FLOW OF HEAT, I

7.1. In Chaps. 3 to 5 we have already discussed a number of the simpler problems of heat flow. These have included the case of the steady state for several different conditions, and the simplest case in which the temperature varies with time, *viz.*, the periodic flow. With the single exception of the steady state for a plane, in which we were forced to assume one of the results derived later in the study of Fourier's series, these problems could all be solved without the use of this analysis; but we now come to a class of problems, at once more interesting and more difficult, in which continual use is made of Fourier's series and integrals.

In the present chapter and the following one we shall take up a number of cases of the flow of heat in one dimension. These will include the problem of the infinite solid, in which the heat is supposed to have a given initial distribution—*i.e.*, the initial temperature is known for every point—and starts to flow at time $t = 0$; the so-called "semiinfinite solid" that has one plane bounding face, usually under a given condition of temperature; the slab with its two plane bounding faces; also, the case of the long rod with radiating surface; and the problem of heat sources. In these several cases the solutions hold equally well for the one-dimensional flow of heat in an infinite solid, or for the flow along a rod whose surface, save in the fourth case above mentioned, is supposed to be impervious to heat. In all the problems discussed in this chapter, save that of the radiating rod, the solutions must first of all satisfy the Fourier conduction equation, which becomes for one dimension

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \qquad (a)$$

As we saw in Sec. 3.5, this must be modified for the case of the radiating rod by the addition of a third term.

CASE I. INFINITE SOLID. INITIAL TEMPERATURE DISTRIBUTION GIVEN

7.2. Take the x direction as that of the flow of heat. Then, all planes parallel to the yz plane will be isothermal surfaces, and the initial temperature of these surfaces is given as a function of their x coordinates. The problem is to determine their temperatures at any subsequent time.

The solution must satisfy (7.1a) and the condition

$$T = f(x) \qquad \text{when } t = 0 \tag{a}$$

We shall solve (7.1a) by a process that is, at the outset, the same as that employed in Sec. 5.2, *viz.*, the substitution in (7.1a) of

$$T = e^{bt+cx} \tag{b}$$

b and c being parameters. This gives

$$b = \alpha c^2 \tag{c}$$

Putting now

$$c = \pm i\gamma \tag{d}$$

instead of $b = \pm i\gamma$ as before, we get

$$T = Le^{-\alpha\gamma^2 t}e^{i\gamma x} \tag{e}$$

and

$$T = Me^{-\alpha\gamma^2 t}e^{-i\gamma x} \tag{f}$$

But since

$$e^{\pm i\gamma x} = \cos\gamma x \pm i \sin\gamma x \tag{g}$$

we get, on combination of (e) and (f) by addition or subtraction—choosing suitable values for L and M—the particular solutions

$$T = e^{-\alpha\gamma^2 t}\cos\gamma x \tag{h}$$

and

$$T = e^{-\alpha\gamma^2 t}\sin\gamma x \tag{i}$$

These are particular solutions of (7.1a) for any value of γ, the latter being a function of neither x nor t. Now we can multiply these by B and C, any functions of γ, and obtain the sum of an infinite series of terms represented by

$$T = \int_0^\infty (B\cos\gamma x + C\sin\gamma x)e^{-\alpha\gamma^2 t}\,d\gamma \tag{j}$$

also as a solution of (7.1a) by the proposition of Sec. 2.4.

The functions B and C must be so determined that for $t = 0$ (j) becomes equal to $f(x)$. Now Fourier's integral (6.10g) gives

$$f(x) = \frac{1}{\pi}\int_0^\infty d\gamma \int_{-\infty}^\infty f(\lambda)\cos\gamma(\lambda - x)\,d\lambda \qquad (k)$$

and from (j) this must equal

$$\int_0^\infty (B\cos\gamma x + C\sin\gamma x)\,d\gamma \qquad (l)$$

Hence, $$B = \frac{1}{\pi}\int_{-\infty}^\infty f(\lambda)\cos\gamma\lambda\,d\lambda \qquad (m)$$

and $$C = \frac{1}{\pi}\int_{-\infty}^\infty f(\lambda \;\sin\gamma\lambda\,d\lambda \qquad (n)$$

and if these values are substituted in (j), we finally have

$$T = \frac{1}{\pi}\int_0^\infty e^{-\alpha\gamma^2 t}\,d\gamma \int_{-\infty}^\infty f(\lambda)\cos\gamma(\lambda - x)\,d\lambda \qquad (o)$$

This is then the required solution, for it satisfies (7.1a) and reduces for $t = 0$ to (k), *i.e.*, to $f(x)$. It gives the value of T for any chosen values of x or t.

7.3. This equation can be simplified and put in a more useful form by changing the order of integration and evaluating one of the integrals. For

$$T = \frac{1}{\pi}\int_{-\infty}^\infty f(\lambda)\,d\lambda \int_0^\infty e^{-\alpha\gamma^2 t}\cos\gamma(\lambda - x)\,d\gamma \qquad (a)$$

But since (see Appendix C)

$$\int_0^\infty e^{-m^2y^2}\cos ny\,dy = \frac{\sqrt{\pi}}{2m}\,e^{-n^2/4m^2} \qquad (b)$$

we have

$$\int_0^\infty e^{-\alpha\gamma^2 t}\cos\gamma(\lambda - x)\,d\gamma = \frac{1}{2}\sqrt{\frac{\pi}{\alpha t}}\,e^{-(\lambda-x)^2/4\alpha t} = \eta\sqrt{\pi}\,e^{-(\lambda-x)^2\eta^2} \qquad (c)$$

putting $\eta \equiv 1/(2\sqrt{\alpha t})$. Hence

$$T = \frac{\eta}{\sqrt{\pi}}\int_{-\infty}^\infty f(\lambda)e^{-(\lambda-x)^2\eta^2}\,d\lambda \qquad (d)$$

By putting $$\beta \equiv (\lambda - x)\eta \quad \text{or} \quad \lambda = \frac{\beta}{\eta} + x \qquad (e)$$

we secure the still shorter form

$$T = \frac{1}{\sqrt{\pi}}\int_{-\infty}^\infty f\left(\frac{\beta}{\eta} + x\right)e^{-\beta^2}\,d\beta \qquad (f)$$

We may regard this as our final solution, since it is much easier to handle than the other forms. If $f(x) = C$, a constant, then $f\left(\frac{\beta}{\eta} + x\right) = C$, and the integral reduces to the "probability integral" (see Appendix D). If $f(x) = x^2$, say, then the equation (f) becomes

$$T = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left(\frac{\beta^2}{\eta^2} + \frac{2\beta x}{\eta} + x^2\right) e^{-\beta^2}\, d\beta \qquad (g)$$

x being a constant as regards this integration, these three integrals can be readily evaluated (see Appendixes B, C, and D). Also, for many other forms of $f(x)$ the integration is not difficult.

7.4. If $f(x)$ is of more than one form, or possesses discontinuities, it may be necessary to split the integral (7.3f) into two or more parts. For example, suppose that $f(x) = T_0$ between the limits $x = l$ and $x = m$, and that $f(x) = 0$ outside these limits, a condition that would correspond to the sudden introduction of a slab at temperature T_0 between two infinite blocks of the same material and at zero temperature. We write the integral (7.3f)

$$T = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{b} 0 e^{-\beta^2}\, d\beta + \frac{1}{\sqrt{\pi}} \int_{b}^{c} T_0 e^{-\beta^2}\, d\beta + \frac{1}{\sqrt{\pi}} \int_{c}^{\infty} 0 e^{-\beta^2}\, d\beta \qquad (a)$$

In determining the limits b and c it must be remembered that x (as well as t) is a constant for each particular evaluation of the integral, and that the initial temperature condition is really expressed as a function of the variable of integration λ, *i.e.*, $T_0 = f(\lambda)$. The limits of b and c will then be the values of β corresponding to $\lambda = l$ and $\lambda = m$; and from (7.3e) these are seen to be $(l - x)\eta$ and $(m - x)\eta$, respectively. Equation (a) then reduces to

$$T = \frac{T_0}{\sqrt{\pi}} \int_{(l-x)\eta}^{(m-x)\eta} e^{-\beta^2}\, d\beta \qquad (b)$$

This solution may be readily applied to the case in which $f(x) = T_0$ for $x > 0$, and $f(x) = 0$ for $x < 0$, for in this event the limits are seen at once to be $-x\eta$ and ∞.

APPLICATIONS

7.5. Concrete Wall. While perhaps not having the variety of applications that we shall find for Case II, next to be considered, the foregoing equations admit of the solution of many interesting problems. For example, suppose a concrete wall 60 cm (23.6 in.) thick is to be formed by pouring concrete in a trench cut in soil at a temperature of $-4°C$ (24.8°F), the concrete being poured at 8°C (46.4°F). It is desired to know how long it will be before the freezing temperature will penetrate the wall to a depth of 5 cm (2 in.). In other words, will the wall as a whole have time to "set" before it is frozen?

To apply the foregoing equations we must first assume that the soil has the same diffusivity (we shall use $\alpha = 0.0058$ cgs) as the concrete, as would be approximately true in many cases, and that latent-heat considerations can be neglected. The solution then follows at once from the equation of the last section. Taking the origin at the center of the wall, we have $l = -30$ cm, $m = 30$ cm, and $x = \pm 25$ cm. Choosing, say, the positive value for x, and shifting our temperature scale so that the initial soil temperature is brought to zero, while the freezing temperature becomes 4°C and the initial wall temperature 12°C, (7.4*b*) becomes

$$4 = \frac{12}{\sqrt{\pi}} \int_{-55\eta}^{5\eta} e^{-\beta^2}\, d\beta \tag{a}$$

To find t we must determine the limit p $(\equiv 5\eta)$ so that

$$\frac{2}{\sqrt{\pi}} \int_{-11p}^{p} e^{-\beta^2}\, d\beta \left(= \frac{2}{\sqrt{\pi}} \int_0^p e^{-\beta^2}\, d\beta + \frac{2}{\sqrt{\pi}} \int_0^{11p} e^{-\beta^2}\, d\beta \right) = \frac{2}{3} \tag{b}$$

From the probability-integral table (Appendix D) we readily find p to be about 0.055, or $\eta = 0.011$, which gives

$$t \left(= \frac{1}{4\alpha\eta^2} \right) = 356{,}000 \text{ sec} = 4.1 \text{ days} \tag{c}$$

If we are interested in knowing the temperature at the center of the wall at the end of this 4.1-day interval, we put $t = 356{,}000$ sec (*i.e.*, $\eta = 0.011$) in the equation

$$T = \frac{12}{\sqrt{\pi}} \int_{-30\eta}^{30\eta} e^{-\beta^2} d\beta = 4.31°C \qquad (d)$$

Subtracting the 4°C that was added to shift the temperature scale so as to make the initial temperature of the soil zero, we have $T_c = 0.31°C$. This indicates that the whole wall is near the freezing point.

7.6. It may be remarked that in solving this problem we have also accomplished the solution of another that, at first sight, appears by no means identical with it. Suppose the same temperature conditions to exist, but the wall to be only half as thick, and one face in contact, not with earth, but with some material practically impervious to heat, or at least a very much poorer conductor than cement; *e.g.*, cork or concrete forms of dry wood. To see the similarity of the two problems, notice that in the first one conditions of symmetry* show that there would be no transference of heat across a middle plane in the wall; hence, this plane could be made of material impervious to heat without altering the conditions. We could then remove half of the wall without affecting the half on the other side of this impermeable plane, in which case we should have our present problem.

7.7. In the above solutions we have omitted consideration of three important factors which would generally be present in any practical case, and which would serve to retard to a considerable extent the freezing of the wall. These are the latent heat of freezing of the water of the concrete, the heat of reaction that accompanies the setting of concrete, and the insulating effect of wooden forms that are frequently used for such a wall. The theoretical treatment of these factors would be beyond the aims of the present work.

7.8. Thermit Welding. As a further application let us take another and more difficult problem. Suppose two sections of a steel ($\alpha = 0.121$ cgs) shaft 30 cm (11.8 in.) in diameter are to be welded end to end by the thermit process. The crevice between the ends is 8 cm (3.1 in.) wide, and the pouring temperature of the molten steel is assumed to be about 3000°C,

* It is to be noted that this point of view demands a temperature condition symmetrical about the middle plane of the wall. That this is satisfied in the present case, *i.e.*, $f(\lambda) = T_0$, a constant, is evident.

while the shaft is heated to 500°C (*i.e.*, some preheating). It is found that a temperature much above 700°C (the "recalescence point") modifies to some extent the character of the steel of the shaft, and it is desired to know, then, to what depth this temperature will penetrate, or, in other words, how far back from the ends this overheating will extend.

We shall attempt only an approximate solution of this problem, neglecting any changes that the thermal constants undergo at higher temperatures, also radiation losses and other complicating factors, and shall interpret it as that of the introduction of a "slab" of steel at 3000°C between two infinite masses of steel at 500°C. Taking the origin in the middle and putting $l = -4$ and $m = 4$, (7.4*b*) becomes, after shifting the temperature scale 500°C,

$$200 = \frac{2{,}500}{\sqrt{\pi}} \int_{(-4-x)\eta}^{(4-x)\eta} e^{-\beta^2}\, d\beta \qquad (a)$$

Our problem is then to find the largest value of x that will satisfy the above relation, *i.e.*, that will afford a value of the above integral equal to $^{200}\!/_{1250}$, or 0.16.

We can most conveniently arrive at a solution by the method of trial and error. Thus, if $x = 5$, *i.e.*, 1 cm from the original end of the shaft, the limits of the above integral may be called -9η and $-\eta$, and a little inspection of the table in Appendix D shows that to give the integral the value 0.16, η must be either 0.018 or 0.994. For $x = 10$ the limits are -14η and -6η, which necessitates η being either 0.019 or 0.165; and a few more trials show that if $x = 24.3$, with corresponding limits of -28.3η and -20.3η, there is only a single value to be found for η, and this is approximately equal to 0.029.

This, then, is the key to the solution, for the second and larger of the two η values in the above pairs will evidently give the shorter time, or, in other words, the time at which the point first reaches this temperature. For the smaller values of x the temperature goes higher than this value of 700°C and later falls to this point at a time afforded by the first value of η. When the two values are just equal, it means that the temperature just reaches this value, and the time in this case will be given by

$$t = \frac{1}{4\alpha\eta^2} = \frac{1}{4 \times 0.121 \times 0.029^2} = 2{,}460 \text{ sec} \tag{b}$$

The overheating then extends in to 20.3 cm (8.0 in.) from the end and reaches this point in 41 min.*

7.9. It is well to note in these, as in any other applications, how the results would be affected by changes in the conditions that enter. In the first case, for instance, it is readily seen that the time will come out the same for any two temperatures of the soil and concrete that have the same ratio; *e.g.*, -2 and $+4°$, or -15 and $+30°$. Moreover, a consideration of the limits shows that the time is inversely proportional to the diffusivity α. In the last illustration this same inverse proportionality of time and diffusivity also holds, and we can in addition draw the rather striking conclusion that the *depth* to which a given temperature will penetrate under such conditions is independent of the thermal constants of the medium.† The *time* it takes to reach this depth however, depends, as just mentioned, on the diffusivity.

7.10. Problems

1. Show that if the initial temperature is everywhere T_0, a constant, the temperature must always be T_0.

In this case
$$T = \frac{T_0}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\beta^2}\, d\beta = T_0 \tag{a}$$

(See Appendix D for values of the probability integral.)

2. Show that, if T is initially equal to x, it must always be equal to x; and, if it is initially equal to x^2, it will be $x^2 + 2\alpha t$ at any time later.

3. In the application of Sec. 7.5 determine when the freezing temperature will reach the center of the wall. *Ans* 4.8 days

4. A slab of molten lava at 1000°C and 40 m thick is intruded in the midst of rock at 0°C. What will be the temperatures at the center and sides of the slab after cooling for 1 day and for 100 years? Use $\alpha = 0.0118$ for both lava and surrounding rock.

Ans. Center, 1000°C and 183°C; sides, 500°C and 178°C

* It is obvious that a more exact solution of this problem might be obtained by a process of differentiation. This is left as an exercise for the ambitious reader.

† This is only true, of course, when the heated material introduced is of the same character as the body itself.

5. Frozen soil at $-6°C$ is to be thawed by spreading over the surface a 15-cm layer of hot ashes and cinders at 800°C and then covering the surface of this layer with insulating material to prevent heat loss. Taking the diffusivity of soil and ashes as 0.0049 cgs and assuming that the latent heat of fusion of the water content may be taken account of by supposing that the soil has to be raised to, say, 5°C instead of merely to zero, to produce melting, how far will the thawing proceed in half a day?

SUGGESTION: Try $x = 50$ cm, 60 cm, etc. Note that the problem is equivalent to that for a slab of twice the thickness with ground on each side.

Ans. 45 cm, or $x = 60$ cm

6. A metal bar ($\alpha = 0.173$ cgs) l cm long, in which the temperatures have reached a steady state with one end at 0°C and the other at 100°C, is placed in end-to-end contact between two very long similar bars at 0°C. Assuming that the surfaces of the bars are insulated to prevent loss of heat, and taking the origin at the zero end of the middle bar, work out the formula for the temperature at any point and apply it to a bar 100 cm long after 15 min of cooling. Find the temperatures at the center, at the hot end, and at the cold end. *Ans.* 49.75°C, 42.95°C, 7.05°C

7. A great pile of soil ($\alpha = 0.0031$ cgs) at $-30°C$ is deposited on similar soil at $+2°C$. Latent-heat considerations neglected, how long will it take the zero temperature to penetrate to a depth of 1 m? *Ans.* 7.9 days

8. In the application of Sec. 7.8 compute the distance to which the temperature 1300°C will penetrate. *Ans.* 2 cm

CASE II. SEMIINFINITE SOLID WITH ONE PLANE BOUNDING FACE AT CONSTANT TEMPERATURE. INITIAL TEMPERATURE DISTRIBUTION GIVEN

7.11. This is the case of the body extending to infinity in the positive x direction only, and bounded by the yz plane, which is kept at a constant temperature. The temperature for every point (plane) of the body is given for the time $t = 0$.

7.12. Boundary at Zero Temperature. We have here to seek a solution of

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \tag{7.1a}$$

subject to the conditions $\quad T = 0 \qquad$ at $x = 0 \qquad (a)$

and $\quad T = f(x) \qquad$ when $t = 0 \qquad (b)$

It is possible to treat this as a special form of Case I (Secs. 7.2 to 7.4) by imagining that for every positive (or negative) temperature at distance x there is an equal negative (or positive)

temperature at distance $-x$. In other words, if there should be a distribution of heat on the side of the negative x identical with, but opposite in sign to, that on the positive side, the flow of heat would be such as to keep the temperature of the yz plane continually zero. A little thought on the symmetry of such a temperature distribution will suffice to show that this conclusion is sound; for there is no more reason for the boundary surface to take positive temperatures under these conditions than negative, and hence its temperature will be zero.

To express this condition mathematically, let us suppose that for points on the positive side of the origin $\lambda = \lambda_1$, and on the negative side $\lambda = -\lambda_2$. Then, λ_1 and λ_2 are each essentially positive, and the temperature $f(\lambda)$ can be expressed as $f(\lambda_1)$ for the positive region and $-f(\lambda_2)$ for the negative. Equation (7.3d) can then be written for this case

$$T = \frac{\eta}{\sqrt{\pi}}\left[\int_0^\infty f(\lambda_1)e^{-(\lambda_1-x)^2\eta^2}\,d\lambda_1 + \int_\infty^0 -f(\lambda_2)e^{-(-\lambda_2-x)^2\eta^2}\,(-d\lambda_2)\right] \qquad (c)$$

the lower limit of the second integral being $+\infty$ instead of $-\infty$, as it would be if λ were the variable. But since the value of a definite integral is independent of the variable of integration (cf. Secs. 6.7 and 6.10), we can substitute λ (or any other symbol) for λ_1 and λ_2 in the above equation, which can then be reduced to

$$T = \frac{\eta}{\sqrt{\pi}}\int_0^\infty f(\lambda)\,[e^{-(\lambda-x)^2\eta^2} - e^{-(\lambda+x)^2\eta^2}]\,d\lambda \qquad (d)$$

Making substitutions similar to (7.3e), *viz.*,

$$\beta \equiv (\lambda - x)\eta \qquad \beta' \equiv (\lambda + x)\eta \qquad (e)$$

this becomes

$$T = \frac{1}{\sqrt{\pi}}\left[\int_{-x\eta}^\infty f\left(\frac{\beta}{\eta} + x\right)e^{-\beta^2}\,d\beta - \int_{x\eta}^\infty f\left(\frac{\beta'}{\eta} - x\right)e^{-\beta'^2}\,d\beta'\right] \qquad (f)$$

or, what amounts to the same thing,

$$T = \frac{1}{\sqrt{\pi}}\left[\int_{-x\eta}^\infty f\left(\frac{\beta}{\eta} + x\right)e^{-\beta^2}\,d\beta - \int_{x\eta}^\infty f\left(\frac{\beta}{\eta} - x\right)e^{-\beta^2}\,d\beta\right] \qquad (g)$$

It is well to assure ourselves that (g) is the required solution. From the manner of its formation, *i.e.*, originally from (7.2h) and (7.2i), it must be a solution of (7.1a), while for $x = 0$ the two integrals are evidently equal and opposite in sign; thus, condition (a) is fulfilled. As to condition (b) we see that for $t = 0$ the second integral vanishes, and the whole expression reduces to

$$T = f(x)\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\beta^2}\,d\beta = f(x) \tag{h}$$

7.13. Surface at Zero; Initial Temperature of Body T_0. An interesting special case is that in which the initial temperature is T_0 throughout the body except at the yz surface, which is kept at zero. $f(\lambda)\left[= f\left(\frac{\beta}{\eta} + x\right) \text{ or } f\left(\frac{\beta}{\eta} - x\right)\right]$ then reduces to T_0, so that (7.12g) becomes

$$T = \frac{T_0}{\sqrt{\pi}}\left(\int_{-x\eta}^{\infty} e^{-\beta^2}\,d\beta - \int_{x\eta}^{\infty} e^{-\beta^2}\,d\beta\right) \tag{a}$$

$$= \frac{T_0}{\sqrt{\pi}}\int_{-x\eta}^{x\eta} e^{-\beta^2}\,d\beta \tag{b}$$

$$= \frac{2T_0}{\sqrt{\pi}}\int_{0}^{x\eta} e^{-\beta^2}\,d\beta \tag{c}$$

since $e^{-\beta^2}$ is an even function (Sec. 6.10). Equation (c) will be commonly written

$$T = T_0\Phi(x\eta) \tag{d}$$

7.14. Surface at T_s; Initial Temperature of Body Zero. By an extension of (7.13d) we can handle this case at once. For if (7.13d) is written for a negative initial temperature $-T_s$, we have

$$T_1 = -T_s\Phi(x\eta) \tag{a}$$

and, if T_s is then added to each side, we get

$$T = T_1 + T_s = T_s[1 - \Phi(x\eta)]^* \tag{b}$$

* Those familiar with electric circuit theory will recognize that, for $T_s = 1$, T is a sort of "indicial temperature," corresponding to the "indicial voltage" at a point in a circuit due to unit voltage applied at the terminals.

This process is, of course, merely equivalent to shifting the temperature scale, as we have had frequent occasion to do in previous problems.

We can replace (*b*) and (7.13*d*) by a single equation of more general usefulness than either, which applies to a body initially

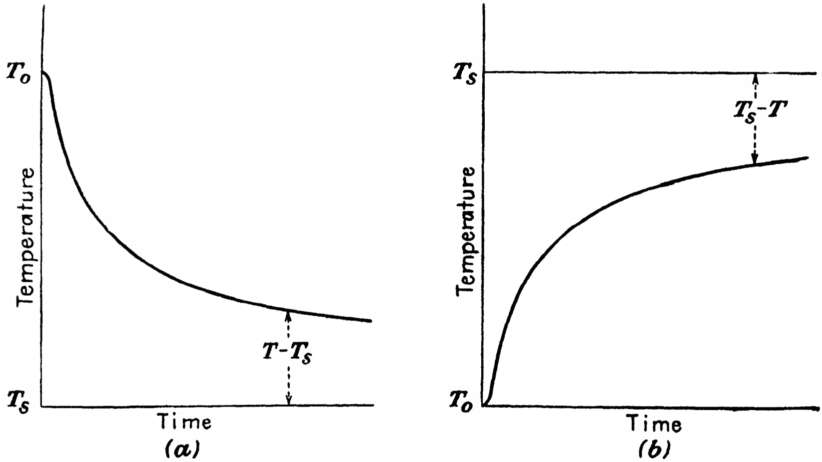

FIG. 7.1.—Cooling and heating curves.

at T_0 and with surface at T_s. Write (7.13*d*) for a surface temperature T_s different from zero, *i.e.*, shift the temperature scale. Then

$$T - T_s = (T_0 - T_s)\Phi(x\eta) \tag{c}$$

or

$$\frac{T - T_s}{T_0 - T_s} = \Phi(x\eta) \tag{d}$$

This holds for either heating or cooling of the body. The quantity $\dfrac{T - T_s}{T_0 - T_s}\left(= \dfrac{T_s - T}{T_s - T_0}\right)$ is readily visualized from Fig. 7.1 as the fraction, at any time t, of the maximum temperature change that still remains to be completed. It is sometimes useful to think of this as a new temperature scale that is independent of the magnitude of the degree in the scale used for T_0 and T_s.

7.15. Law of Times. An interesting fact can be deduced from (7.13*c*) and (7.14*d*), for it is easily seen that any particular temperature T is attained at distances x_1 and x_2 from the bound-

ary surface in times t_1 and t_2 conditioned by the relation

$$\frac{x_1}{2\sqrt{\alpha t_1}} = \frac{x_2}{2\sqrt{\alpha t_2}} \tag{a}$$

or

$$\frac{t_1}{t_2} = \frac{x_1^2}{x_2^2} \tag{b}$$

This gives the law that *the times required for any two points to reach the same temperature are proportional to the squares of their distances from the boundary plane,* a statement that is true whether the body is initially at a uniform temperature and the surface at zero, or initially at zero and the surface heated, provided only that the surface keeps its temperature constant in each case.

It can also be at once deduced that the time required for any point to reach a given temperature is inversely proportional to the diffusivity α. Both these relations are of wide application, and the one or the other of them holds good for a large number of cases of heat conduction. We have already noted a case in which the second law holds in Sec. 7.9.

7.16. Rate of Flow of Heat. We can now determine the rate at which heat flows into or out of a body, initially at T_0 and with surface at T_s, through any unit of area of plane surface parallel to the boundary. To do this differentiate (7.14*c*), using Appendix K. Then,

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial (x\eta)} \frac{\partial (x\eta)}{\partial x} = \frac{2(T_0 - T_s)\eta}{\sqrt{\pi}} e^{-x^2\eta^2} \tag{a}$$

The rate of flow of heat into the body through any unit area parallel to the yz boundary plane is then

$$w = -k\frac{\partial T}{\partial x} = \frac{2k(T_s - T_0)\eta}{\sqrt{\pi}} e^{-x^2\eta^2} \tag{b}$$

or for the boundary plane $x = 0$

$$w_s = \frac{2k(T_s - T_0)\eta}{\sqrt{\pi}} = \frac{k(T_s - T_0)}{\sqrt{\pi\alpha t}} \tag{c}$$

To get the total heat inflow at the surface between times t_1

and t_2 we integrate (c) and get

$$\frac{Q}{A} = \int_{t_1}^{t_2} w_s\, dt = \frac{2k(T_s - T_0)}{\sqrt{\pi\alpha}} (\sqrt{t_2} - \sqrt{t_1}) \tag{d}$$

7.17. Temperature of Surface of Contact. Suppose two infinite bodies B and C of conductivities and diffusivities k_1, α_1, and k_2, α_2, respectively, each with a single plane surface and with these surfaces placed in contact. Assume that B and C are initially at temperatures T_1 and T_2, respectively, and imagine for the moment that the boundary surface is kept, either by the continuous addition or subtraction of heat, at the constant temperature T_s, where $T_1 > T_s > T_2$. We shall determine what conditions must be fulfilled that this surface of contact may receive as much heat from one body as it loses to the other and hence will require no gain or loss of heat from the outside to keep constantly at T_s; in other words, we shall determine this temperature of the surface of contact.

Each unit of area of surface of contact receives heat from B at the rate [see (7.16c)]

$$w_1 = \frac{k_1(T_1 - T_s)}{\sqrt{\pi\alpha_1 t}} \tag{a}$$

while it loses to C at the rate

$$w_2 = \frac{k_2(T_s - T_2)}{\sqrt{\pi\alpha_2 t}} \tag{b}$$

Then, if these two are equal, the boundary plane will neither gain nor lose heat permanently and hence will remain constant in temperature. Thus,

$$\frac{k_1(T_1 - T_s)}{\sqrt{\alpha_1}} = \frac{k_2(T_s - T_2)}{\sqrt{\alpha_2}} \tag{c}$$

or

$$T_s = \frac{k_1T_1/\sqrt{\alpha_1} + k_2T_2/\sqrt{\alpha_2}}{k_1/\sqrt{\alpha_1} + k_2\sqrt{\alpha_2}} \tag{d}$$

If $k_1 = k_2$ and $\alpha_1 = \alpha_2$, $T_s = (T_1 + T_2)/2$, as we should expect. The same holds if $k_1c_1\rho_1 = k_2c_2\rho_2$.

APPLICATIONS

7.18. Concrete. In a fire test the surface of a large mass of concrete ($\alpha = 0.030$ fph) was heated to 900°F; how long should it take the temperature 212°F to penetrate 1 ft if the initial temperature of the mass was 70°F?

From (7.14*d*) we have

$$\frac{212 - 900}{70 - 900} = \Phi(x\eta) = \Phi\left(\frac{2.89}{\sqrt{t}}\right) \tag{a}$$

from which we get, using Appendix D, $t = 8.9$ hr.

7.19. Soil. How far will the freezing temperature penetrate in 24 hr in soil ($\alpha = 0.0049$ cgs) at 5°C if the surface is lowered to -10°C?

Using (7.14*d*), $$\frac{0 + 10}{5 + 10} = \Phi(x\eta) = \Phi\left(\frac{x}{41.1}\right) \tag{a}$$

from which we get $x = 28.2$ cm. For twice this depth it would take 4 days, three times, 9 days, etc.

If the initial temperature of soil is 2°C (35.6°F) and the surface is cooled to -24°C (-11°F), how long will it be before the temperature will fall to zero at the depth of 1 m?

$$24/26 = \Phi(x\eta); \qquad t = 326{,}000 \text{ sec} = 3.8 \text{ days} \tag{b}$$

Since no account has been taken of the latent heat of freezing for the moisture of the soil in the last two problems, the distance in the first problem is undoubtedly too large, and the time in the second too small, for the actual case. Even in the case of concrete, unless it is old and thoroughly dry, there is a considerable lag in the heating effect as the boiling point is passed, showing latent-heat effects.

An exact treatment of these latent-heat considerations must be reserved for Chap. 10, but in the following problem an approximate solution for a particular case is suggested.

7.20. The Thawing of Frozen Soil. Soil at -6°C (21°F), of diffusivity 0.0049 cgs and moisture content 3 per cent, is to be thawed by heating the surface with a coke fire to 800°C (1472°F). The question is: How far will the thawing proceed in a given time?

To take account of the latent heat of fusion of the 3 per cent moisture we note that, since the specific heat of such soil is taken as 0.45 (undoubtedly, however, this is a rather high figure for such small moisture content), the heat required to thaw this moisture per gram of soil would be the same as that which would raise this soil $0.03 \times 80 \div 0.45$, or about 5°C in temperature. This is nearly equivalent to saying that the soil must be raised to 5°C (41°F) to produce thawing, *i.e.*, a total rise of 11°C. Then,

$$11 = 806[1 - \Phi(x\eta)] \tag{a}$$

and we find that $x\eta \equiv x/(2\sqrt{\alpha t})$ must be about 1.74, or

$$t = \frac{x^2}{0.0595} = 16.8\, x^2 \tag{b}$$

Then for a thawing of 45 cm (1.5 ft), $t = 34{,}000$ sec, or 9.5 hr; and for 90 cm (3 ft) 38 hr, etc.

While local conditions (varying diffusivities and moisture contents) would alter these figures considerably, the law that the time for thawing would vary as the square of the depth holds good in any case in which the soil is initially at sensibly the same temperature throughout. If it is not as cold below, the thawing will proceed faster than this law would indicate.

7.21. Shrink Fittings. As a problem of a somewhat different type from the preceding let us consider the thermal principles involved in the removal by heating of a ring or collar that has been shrunk on to a cylinder or wheel. If the thickness is small compared with the diameter, it may be treated as a case of one-dimensional transmission, and as a very good example we may cite the case of the locomotive tire. Suppose such a tire 7.62 cm (3 in.) thick is to be removed by heating its outer surface; let us question at what time the differential expansion of tire and rim would be a maximum and hence the tire be most readily removed. We shall assume that this differential expansion is determined by the magnitude of the temperature gradient across the boundary of tire and rim. From (7.16*a*), putting $T_0 = 0$,

$$\frac{\partial T}{\partial x} = -\frac{T_s}{\sqrt{\pi \alpha t}}\, e^{-x^2/4\alpha t} \tag{a}$$

To find when this is a maximum, differentiate with respect to t and equate to zero. Then,

$$t = \frac{1}{2}\frac{x^2}{\alpha} \tag{b}$$

$$\therefore \left(\frac{\partial T}{\partial x}\right)_{\max} = \frac{-T_s}{\sqrt{\pi \alpha t e}} \tag{c}$$

So in this case ($\alpha = 0.121$ cgs), $t = 240$ sec, or 4 min.

The above discussion of the problem is based on the conditions of Sec. 7.14, *viz.*, for the surface heated suddenly to the

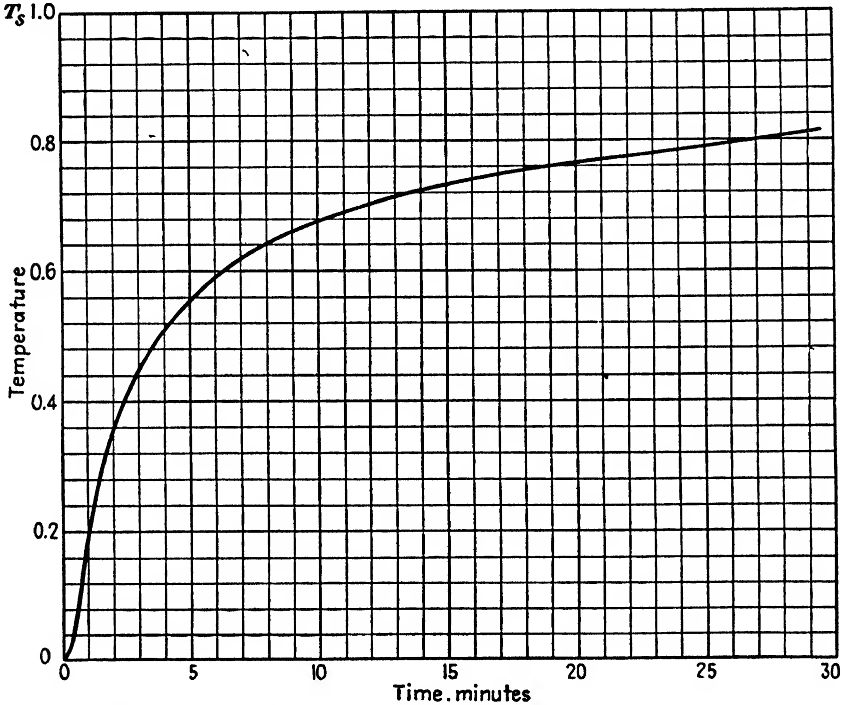

FIG. 7.2. A type of theoretical temperature-time curve obtained on the assumptions of Sec. 7.21. (The more nearly the actual heating curve of the surface approaches this type, the better the case can be handled theoretically.)

temperature T_s, as by immersion in a bath of molten metal. As a matter of fact, the surface heating in the practical case would generally be a more gradual process, brought about in many cases by a gas flame. A rigorous solution of this complicated problem is very difficult, but the following is offered as being a

good approximate solution. Imagine in the case of the locomotive tire just considered that 5 cm thickness is added to the tire and that the outer surface is, as before, suddenly raised to temperature T_s. The temperature of the original surface will then be given by (7.14b) and will be found to rise gradually (see Fig. 7.2), increasing more rapidly at first and more slowly later, just as would be the case if this surface were flame heated. By varying the thickness of metal that we are to assume added (the 5 cm added in this case yields a very plausible curve) and plotting the temperature-time curve as in Fig. 7.2 for each case, a result may be obtained very nearly like the actual heating conditions.*

The problem is then reduced to the preceding, save that the tire is imagined to be 5 cm thicker. The time comes out 11 min. For a slower rate of heating the time would be correspondingly longer.

A point of interest in this connection is a comparison of the actual maximum temperature gradients for the rapid and slow heating, for these are the measure of the ease—or the possibility—of removal of such a shrunk fitting. Putting $t = 240$ sec in (c), we get $(\partial T/\partial x)_{max} = -0.064\ T_s$°C/cm, while for $t = 660$ sec [which is the case for the maximum gradient under the slower heating (see Fig. 7.3)], the gradient is only $-0.038T_s$°C/cm. This shows that when difficulty is expected in the removal of any shrunk-on collar, the surface heating should be done as quickly as possible, perhaps with the use of molten metal or even thermit. The above calculations would also serve to show the time for which it is desirable to continue this heating. From

* The reasoning involved here is as follows: If the outer surface A of this imaginary 12.62-cm (*i.e.*, 7.62 + 5) tire is suddenly heated to T_s, the initial temperature of tire and wheel being zero and the whole treated as a case of one-dimensional flow (which is justifiable since we are concerned with only a relatively small depth below the surface), the temperature of the original surface B will be some $\psi(t)$ as indicated in Fig. 7.2. This may be thought of as a boundary condition for this original boundary B. According to the uniqueness theorem (Sec. 2.6), then, the temperatures inside—*i.e.*, at the "plane" across which we are getting the temperature gradient, where the tire joins the rim—are determined by this $\psi(t)$ irrespective of how it is brought about. It is therefore immaterial whether this $\psi(t)$ is produced by gas heating at the original surface B of the 7.62-cm tire or by a sudden rise of temperature of the surface A of the 12.62-cm tire.

the shape of the curve in Fig. 7.3 it is evident that it is much better to continue the heating too long than to cut it too short.

The considerations of this section would also apply to the so-called "thermal test" of car wheels, which consists in heating the rim of the wheel with molten metal for a given time. The temperature gradient might reasonably be taken as a measure of the stresses introduced in this way, and it could be determined at once from (*a*).

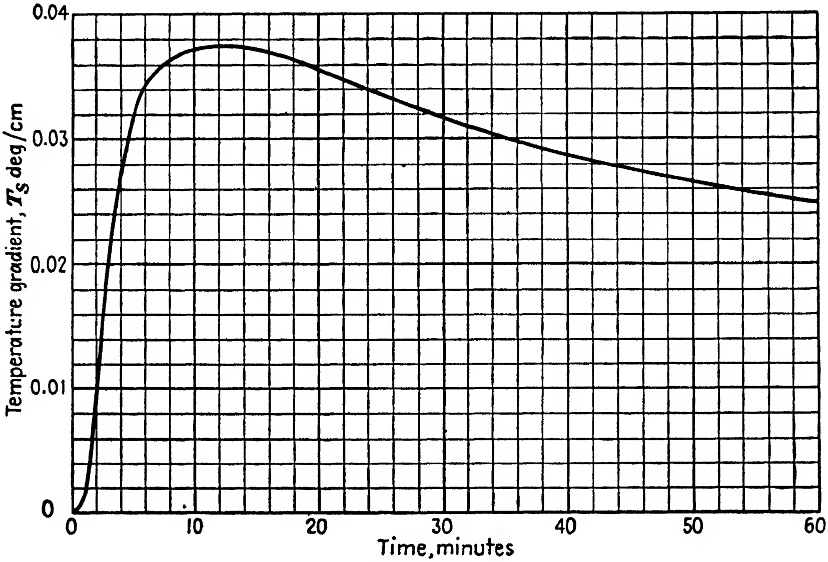

FIG. 7.3. Curve showing the variation of temperature gradient with time, at a distance of 12.6 cm below a surface of steel suddenly heated to T_s; or 7.6 cm below a surface heated as in Fig. 7.2. (The best time to attempt to remove the fitting would be when the gradient—sign is neglected here—is a maximum.)

7.22. Hardening of Steel. A large ingot of steel ($\alpha = 0.121$ cgs) at T_0 has its surface suddenly chilled to T_s. Let us discuss the rate of cooling as a function of the time and of the depth in the metal.

We shall do this by differentiating (7.14*c*) (see also Sec. 7.16), and we find

$$\frac{\partial T}{\partial t} = (T_0 - T_s)\frac{2}{\sqrt{\pi}}e^{-x^2/4\alpha t}\left(\frac{-x}{4t^{3/2}\sqrt{\alpha}}\right) = \frac{(T_s - T_0)xe^{-x^2/4\alpha t}}{2t^{3/2}\sqrt{\pi\alpha}} \qquad (a)$$

which is the formula from which the curves of Fig. 7.4 have

been computed for depths of 0.3 and 1 cm. To apply to a specific problem let us question what the rates of cooling are at these depths if the initial temperature is 800°C (1472°F) and the chilling temperature 20°C (68°F), the times being chosen as those at which the metal is just cooling below the recalescence point (about 700°C or 1292°F).

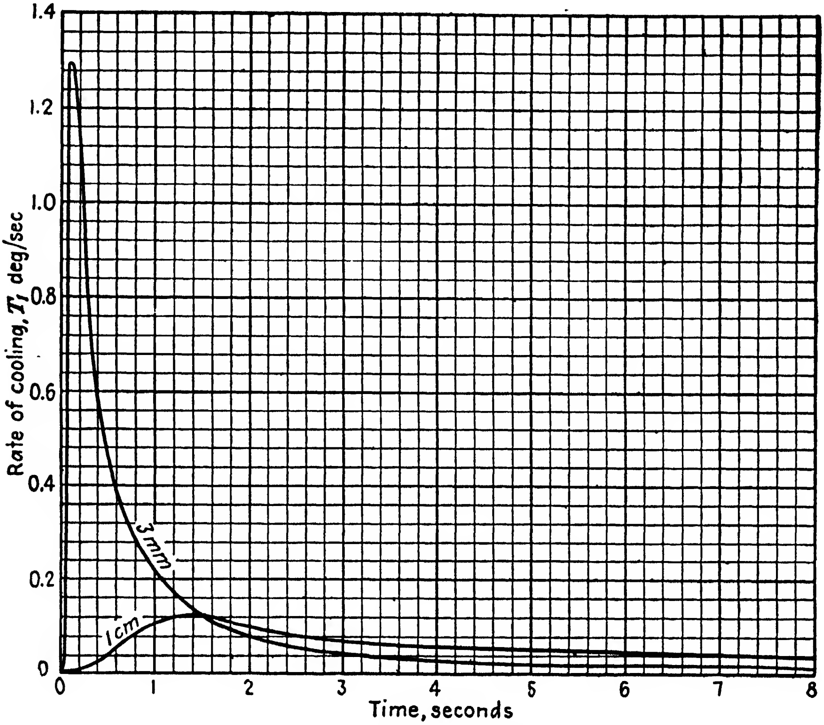

Fig. 7.4. Curves showing rate of cooling at depths of 3 mm and 1 cm below the surface of a steel ingot that is suddenly chilled. T_1 is here $T_0 - T_s$.

To find the times, we put from (7.14*d*)

$$680/780 = \Phi(x\eta) \qquad (b)$$

which gives $t = 0.16$ sec for $x = 0.3$ cm (0.12 in.), and $t = 1.8$ sec for $x = 1$ cm (0.39 in.). From (*a*) or from the curves we then find the rates of cooling to be 920 and 82°C/sec, respectively (1656 and 148°F).

While it might be impossible in practice to attain as sudden a chilling of the surface as the above theory supposes, the curves

of Fig. 7.4 will still serve to give a qualitative explanation of a well-known fact, *viz.*, that the deeper it is desired to have the metal hardened, the hotter it must be before quenching; but that a comparatively small proportional increase in the initial temperature may produce a considerable increase in the depth of the hardening. To explain this it must be noted that one of the factors in hardening is the rate of cooling past the recalescence point. Now from the curves it may be seen that this rate increases to a maximum and then falls off again; hence, for maximum hardness at any given depth the initial temperature should, if possible, be high enough so that the recalescence point will not be passed until the rate of cooling has reached its maximum value.

The rapid chilling of large ingots introduces temperature stresses that frequently result in cracks. Taking the temperature gradient as a measure of this tendency to crack, the subject might be studied theoretically with the equations of the last article.

7.23. Cooling of Lava. We now turn to some applications of a geological nature, the first of which is the cooling of lava under water. Suppose a thickness of, say, 20 m of lava at T_0 (about 1000°C) is flowed over rock at zero and immediately covered with water—perhaps it is ejected under water; what will be its rate of cooling?

Since the water will soon cool the surface at least well below the boiling point, the problem is that of the cooling of a semi-infinite medium with boundary at zero and initial temperature conditions of T_0 as far as $x = l$, and zero from there on to infinity. Formula (7.13a) is for the case where the initial condition is T_0 to infinity, and we may use it by splitting each integral into two, according to the principles explained in Sec. 7.4, the second integral vanishing in each case, since the initial temperature for it would be zero. We have as the formula, then,

$$T = \frac{T_0}{\sqrt{\pi}}\left(\int_{-x\eta}^{(l-x)\eta} e^{-\beta^2}\,d\beta - \int_{x\eta}^{(l+x)\eta} e^{-\beta^2}\,d\beta\right) \qquad (a)$$

Putting Kelvin's value of $\alpha = 0.0118$ cgs for both lava and underlying rock, the accompanying curves (Fig. 7.5) are com-

puted for $l = 20$ m. From the relationship between x and t in the above limits we readily conclude that these same curves apply to a layer n times as thick if the times are taken n^2 times as large, and the distances n times as large.*

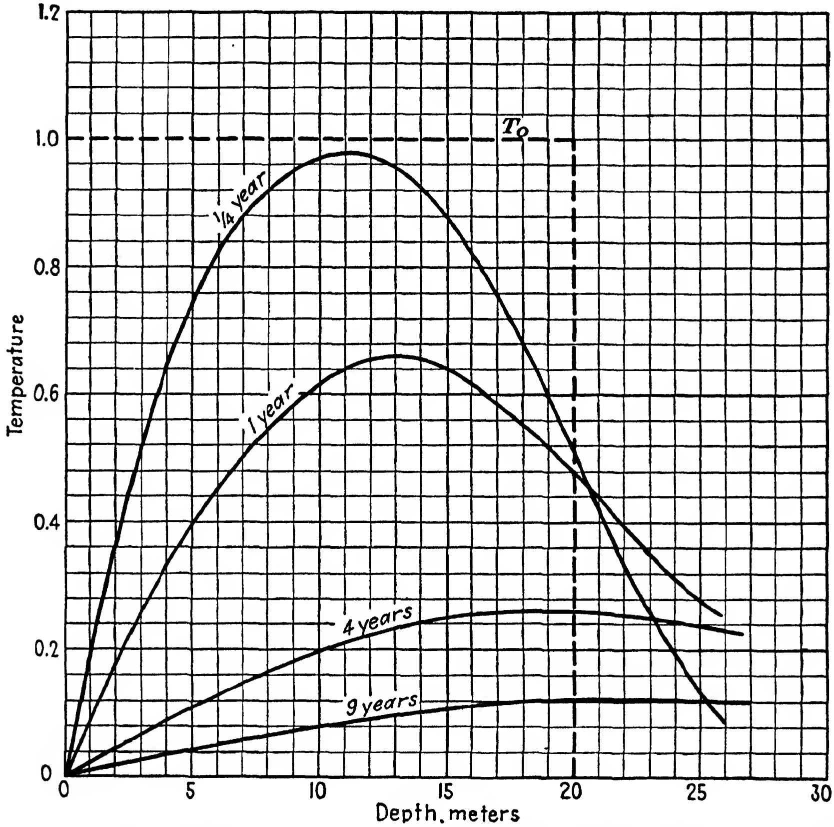

FIG. 7.5. Temperature curves for a layer of lava 20 m thick, after cooling under water for various times.

7.24. The Cooling of the Earth. The problem of the cooling of the earth and the estimate of its age based on such cooling has been discussed by Kelvin† and others‡ as a special case of

* See Boydell,[19] Berry,[13] and Lovering[88] for more extensive treatments of this problem.

† "Mathematical and Physical Papers," III, p. 295; *Smithsonian Report*, 1897, p. 337.

‡ For a good résumé of the subject see Becker,[12] also Slichter,[133] Van Orstrand,[154] and Carslaw and Jaeger.[27a]

the solid with one plane bounding face; for it has been shown that the error introduced in neglecting the curvature is quite negligible. For this purpose the age of the earth is counted from the assumed epoch of Leibnitz's *consistentior status*, when the globe, or rather the crust, had attained a "state of greater consistency" and the formation of the oceans became possible. Kelvin's assumption for this state was an earth whose temperature was in round numbers 3900°C (7000°F.) throughout. He took the average value of the diffusivity as 0.01178 cgs,* and of the present surface gradient of temperature as 1°C in 27.76 m.† The problem is then to find how long it would take for the earth at the assumed initial temperature, and with the surface at a constant temperature approximately zero, to cool until the geothermal gradient at the surface has its present measured value, *viz.*, 1°C in 27.76 m.

Differentiate (7.13c) (see Sec. 7.16). Then,

$$\frac{\partial T}{\partial x} = \frac{2T_0}{\sqrt{\pi}} \frac{e^{-x^2/4\alpha t}}{2\sqrt{\alpha t}} \tag{a}$$

and at $x = 0$

$$\left(\frac{\partial T}{\partial x}\right)_0 = \frac{T_0}{\sqrt{\pi \alpha t}} \tag{b}$$

or

$$t = \frac{T_0^2}{\pi\alpha(\partial T/\partial x)_0^2} \tag{c}$$

Putting in the constants given above, Kelvin got a value of 100 million years for the age of the earth, but because of the uncertainty of the assumptions and data he placed the limits at 20–400 million years, later modifying them to 20–40 million years.

7.25. If the initial temperature of the earth, *i.e.*, its temperature condition at the *consistentior status*, instead of being

Adams,[1] in his discussion of temperatures at moderate depths within the earth, concludes that $\alpha = 0.010$ cgs is the best average for the surface rocks and 0.007 cgs for the deep-seated material.

† Van Orstrand,[153,155] who has made most extensive studies of crustal temperature gradients, places the average for the United States between 1°F in 60 ft and 1°F in 110 ft (1°C in 32.9 and 60.4 m). He states that, for a considerable portion of the sedimentary areas of the globe, an average gradient of 1°F in 50 ft (1°C in 27.4 m) is found either at the surface or at depths of one or two miles.

uniform throughout, increased with the depth, obeying the law*

$$T = f(x) = mx + T_s \tag{a}$$

where T_s is the initial surface temperature and m the initial

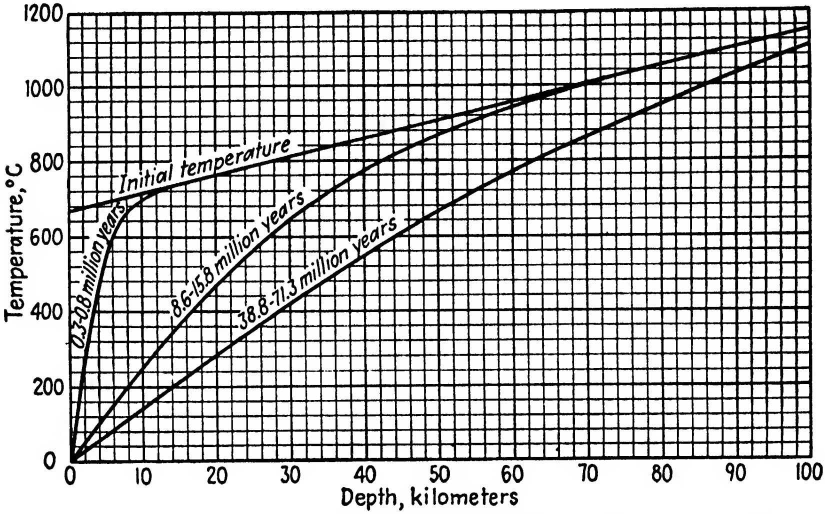

FIG. 7.6. Temperature curves for the earth, after cooling for the specified number of years, assuming the initial conditions of Sec. 7.25. The smaller of the two times is for a diffusivity of 0.0118 cgs (Kelvin), and the larger for 0.0064 cgs. It is to be noted that the temperature state at depths greater than 100 km would be very little affected by cooling for even 50 million years.

gradient, we can solve the problem with the aid of (7.12g); for substitution of (a) in this gives, after some simplification,

$$T = mx + T_s\Phi(x\eta) \tag{b}$$

Differentiating,

$$\frac{\partial T}{\partial x} = m + \frac{T_s}{\sqrt{\pi\alpha t}} e^{-x^2/4\alpha t} \tag{c}$$

or

$$\frac{\pi\alpha}{T_s^2}\left(\frac{\partial T}{\partial x} - m\right)^2 t = e^{-x^2/2\alpha t} \tag{d}$$

When m and x are zero, this reduces, as it should, to Kelvin's solution (7.24c). As it stands, (d) affords a value for the age of the earth, t, in terms of the geothermal gradient $\partial T/\partial x$ at any depth x, under the conditions that the initial temperature of the earth increased uniformly toward the center from some

* Barus.[7]

value T_s at the surface, and that since that time the surface has been kept at the constant temperature zero.

7.26. Effect of Radioactivity on the Cooling of the Earth. Since the discovery of the continuous generation of heat by disintegrating radioactive compounds, much speculation has been indulged in as to the possible effect of such heat on the earth's temperature.* Surface rocks show traces of radioactive materials, and while the quantities thus found are very minute, the aggregate amount is sufficient, if scattered with this density throughout the earth, to supply, many times over, the present yearly loss of heat. In fact, so much heat could be developed in this way that it has been practically necessary to make the assumption that the radioactive materials are limited in occurrence to a surface shell only a few kilometers in thickness.

While a satisfactory mathematical treatment of this problem is impossible with the meager data now available, it can be seen at once that radioactivity would tend to retard the cooling of the earth and hence increase our estimate of its age. A rough idea of the extent to which this is true may be had by assuming that one fourth of the present annual loss of heat is due to this cause, and that the radioactive substances are contained in a very thin outer shell. The geothermal gradient at the bottom of this shell will then be only three fourths of its observed value on the surface, because only three fourths of the heat that passes out from the earth crosses the lower surface. Then, since from (7.25*d*) the age of the earth is inversely proportional to the square of the present gradient at $x = l$, the depth of the radioactive shell (if $m = 0$, and l is small), this would nearly double the calculated age of the earth.

7.27. The Effect of Radioactivity on Earth Temperatures; Mathematical Treatment of a Special Case. While, as remarked above, we know too little of the actual conditions as regards the extent of distribution of radioactive substances in the earth to attempt any rigorous or complete treatment of their effect on the age and temperature of the earth, we can still solve the problem for specially assumed conditions. The assumptions we shall make are at least as consistent as any others with the

* Becker[11] and references in footnotes to Sec. 7.27.

facts as we now know them. The first is that only a fraction, $1/n$, of the total annual heat lost by the earth is due to radioactive causes. The rate of liberation of heat by the disintegration of such substances is supposed to be independent of the time, and the density of distribution of these heat-producing substances is assumed to fall off exponentially with increasing depth below the surface. It was mentioned above that some such assumption as this is practically necessary, for if these substances were scattered throughout the earth with their surface density of distribution, vastly more heat would be generated per year than is actually being conducted through the surface. The second assumption concerns the initial temperature state of the earth; *i.e.*, its temperature distribution at the time of the *consistentior status*. Instead of supposing, as in Kelvin's original calculation, that the earth was at a constant temperature at this time, we shall make the more reasonable assumption of Sec. 7.25, which is based upon data obtained by Barus,* showing the relation of melting point to pressure to be nearly linear for a considerable depth.

In solving the problem we must first modify our fundamental conduction equation so as to take account of this continuous internal generation of heat. We found in Chap. 2 that the rate at which heat is added by conduction to any element of volume $dx\,dy\,dz$ is $k\nabla^2 T\,dx\,dy\,dz$. If in addition heat sources, such as these radioactive products, produce an amount of heat per second represented by $\psi(x,y,z,t)\,dx\,dy\,dz$, then the temperature of this element will be raised at a rate $\partial T/\partial t$ such that

$$k\nabla^2 T\,dx\,dy\,dz + \psi(x,y,z,t)\,dx\,dy\,dz = \frac{\partial T}{\partial t}c\rho\,dx\,dy\,dz \qquad (a)$$

Therefore,

$$\frac{\partial T}{\partial t} = \alpha\nabla^2 T + \frac{\psi(x,y,z,t)}{c\rho} \qquad (b)$$

This is our fundamental equation. For linear flow it takes the form

$$\frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T}{\partial x^2} + \frac{\psi(x,t)}{c\rho} \qquad (c)$$

* See King.[76, p.16]

In the present case the assumption is made that

$$\psi(x,t) = Be^{-bx} \tag{d}$$

where B is the quantity of heat generated per unit volume per second at the surface. Separate determinations of this quantity vary greatly, but the average result will be taken at 0.47×10^{-12} cal/(cm³)(sec) for crustal rocks. The total amount of heat generated in this way per second, and escaping through each square centimeter of the earth's surface, is

$$w_r = \int_0^\infty Be^{-bx}\,dx = \frac{B}{b} \tag{e}$$

But if w_s is the total amount of heat lost by the surface per square centimeter per second,

$$w_r = \frac{w_s}{n} \tag{f}$$

When n is assumed, this enables us to determine b, since both B and w_s are known; *i.e.*,

$$b = \frac{nB}{w_s} \tag{g}$$

Our fundamental equation (c) then becomes

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + Ce^{-bx} \tag{h}$$

where C is written for $B/c\rho$. The solution of this equation must satisfy the boundary conditions

$$T = 0 \qquad \text{at } x = 0 \tag{i}$$
$$T = mx + T_s \qquad \text{when } t = 0 \tag{j}$$

We shall first change (h), by substitution, into a form that is homogeneous and linear. Assume that

$$T = u - \frac{C}{b^2\alpha} e^{-bx} \tag{k}$$

where u is some function of x and t. Then,

$$\frac{\partial T}{\partial t} = \frac{\partial u}{\partial t}; \qquad \frac{\partial^2 T}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} - \frac{C}{\alpha} e^{-bx} \tag{l}$$

and (h) becomes

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \tag{m}$$

The boundary conditions then become

$$u = \frac{C}{b^2\alpha} \qquad \text{at } x = 0 \tag{n}$$

$$u = mx + T_s + \frac{C}{b^2\alpha} e^{-bx} \qquad \text{when } t = 0 \tag{o}$$

Since the problem would be much easier to handle if the first boundary condition were $u = 0$ at $x = 0$, we shall make the further substitution

$$v \equiv u - \frac{C}{b^2\alpha} \tag{p}$$

which gives us, in place of (m),

$$\frac{\partial v}{\partial t} = \alpha \frac{\partial^2 v}{\partial x^2} \tag{q}$$

and for boundary conditions

$$v = 0 \qquad \text{at } x = 0$$

$$v = f(x) = mx + \left(T_s - \frac{C}{b^2\alpha}\right) + \frac{C}{b^2\alpha} e^{-bx} \qquad \text{when } t = 0 \tag{r}$$

This now becomes the problem of Sec. 7.12, where was obtained the solution

$$v = \frac{1}{\sqrt{\pi}}\left[\int_{-x\eta}^{\infty} f\left(\frac{\beta}{\eta} + x\right) e^{-\beta^2}\, d\beta - \int_{x\eta}^{\infty} f\left(\frac{\beta}{\eta} - x\right) e^{-\beta^2}\, d\beta\right] \tag{s}$$

Substituting for $f\left(\frac{\beta}{\eta} + x\right)$ and $f\left(\frac{\beta}{\eta} - x\right)$ from (r), this may be written

$$\begin{aligned} v = {} & \frac{mx}{\sqrt{\pi}}\left(\int_{-x\eta}^{\infty} e^{-\beta^2}\, d\beta + \int_{x\eta}^{\infty} e^{-\beta^2}\, d\beta\right) \\ & + \frac{T_s - C/b^2\alpha}{\sqrt{\pi}}\left(\int_{-x\eta}^{\infty} e^{-\beta^2}\, d\beta - \int_{x\eta}^{\infty} e^{-\beta^2}\, d\beta\right) \\ & + \frac{m}{\sqrt{\pi}\,\eta}\left(\int_{-x\eta}^{\infty} \beta e^{-\beta^2}\, d\beta - \int_{x\eta}^{\infty} \beta e^{-\beta^2}\, d\beta\right) \\ & + \frac{C}{b^2\alpha\sqrt{\pi}}\left(e^{-bx}\int_{-x\eta}^{\infty} e^{\frac{-b\beta}{\eta} - \beta^2}\, d\beta - e^{bx}\int_{x\eta}^{\infty} e^{\frac{-b\beta}{\eta} - \beta^2}\, d\beta\right) \end{aligned} \tag{t}$$

Of the above four terms the first two can readily be shown to equal

$$mx \quad \text{and} \quad \left(T_s - \frac{C}{b^2\alpha}\right)\Phi(x\eta) \tag{u}$$

respectively, while the third vanishes. In evaluating the fourth we note that

$$\int e^{-\frac{b\beta}{\eta}-\beta^2}\,d\beta = e^{\left(\frac{b}{2\eta}\right)^2}\int e^{-\left(\frac{b}{2\eta}+\beta\right)^2}\,d\beta \tag{v}$$

Making use of this fact and of the substitution

$$\gamma \equiv \frac{b}{2\eta} + \beta \tag{w}$$

we have, finally, since

$$\frac{C}{b^2\alpha} = \frac{B}{b^2k} \tag{x}$$

and

$$T = v + \frac{B}{b^2k} - \frac{B}{b^2k}e^{-bx} \tag{y}$$

$$\begin{aligned} T = mx &+ \left(T_s - \frac{B}{b^2k}\right)\Phi(x\eta) \\ &+ \frac{B}{b^2k}\left[1 - e^{-bx} + \frac{e^{\left(\frac{b}{2\eta}\right)^2}}{\sqrt{\pi}}\left(e^{-bx}\int_{\frac{b}{2\eta}-x\eta}^{\infty} e^{-\gamma^2}\,d\gamma \right.\right. \\ &\left.\left. - e^{bx}\int_{\frac{b}{2\eta}+x\eta}^{\infty} e^{-\gamma^2}\,d\gamma\right)\right] \end{aligned} \tag{z}$$

When $B = 0$, *i.e.*, when there is no radioactive material present, this solution reduces, as it should, to (7.25*b*).

A computation of the age of the earth has been made on the basis of (z) for the following assumed conditions:

$$B = 0.47 \times 10^{-12}$$

$w_s = 1.285 \times 10^{-6}$; $n = 4$, *i.e.*, one-fourth of the present heat loss is due to radioactivity; $k = 0.0045$; $c = 0.25$; $\rho = 2.8$; $m = 0.00005$; and $T_s = 995°C$. Then, the time required to cool from the initial conditions* of surface at 995°C and temperature

* Strictly speaking, the conditions are really for a temperature of 1000°C at a depth of 5 km below the surface, the surface itself being, in accordance with the idea of the *consistentior status*, at or near zero in temperature. The above assump-

gradient of 5°C per kilometer to a present surface gradient of 1°C in 35 m comes out to be 45.85×10^6 years. Without radioactivity the same initial conditions give 22.0×10^6 years, so we see that in this case the continuous generation of heat under these conditions increases the computed age of the earth by over 100 per cent.

It may be added that since the estimates of the earth's age based purely on refrigeration are of the same order of magnitude as those arrived at from geological considerations, such as stratigraphy, sodium denudation, etc., some geologists are inclined to believe that radioactivity is not as important in this connection as might be supposed; *i.e.*, that it contributes not more than about one-fourth of the present total annual heat loss. If some such small fraction of the total heat loss is attributed to radioactive causes, estimates of the earth's age based on cooling will be in fair agreement with certain older geological estimates—although far short of the 2×10^9 years which represents the present trend of thought.*

7.28. Problems

1. Show that, under the conditions of Sec. 7.12, if T is initially equal to x, it will always be equal to x; and if it is initially x^2, its value at any time later will be given by

$$2x\sqrt{\frac{\alpha t}{\pi}} \cdot e^{-x^2\eta^2} + (2\alpha t + x^2)\Phi(x\eta)$$

2. If the surface of dry soil ($\alpha = 0.0031$ cgs), initially at 2°C throughout, is lowered to -30°C, how long will it be before the zero temperature will penetrate to a depth of 10 cm? 1 m? (Cf. Problem 7, Sec. 7.10.)

Ans. 77 min; 5.4 days

3. An enormous mass of steel ($k = 0.108$, $\alpha = 0.121$ cgs) at 100°C, with one plane face, is dropped into water at 10°C. Assuming no convection currents in the water (these would be minimized by choosing the face horizontal and on the under side), what will be the temperature of the surface of

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tion of a surface initially at 995°C, which is then suddenly cooled to and thereafter kept at zero, is made to render the problem mathematically simpler. That this would not substantially affect the result may be concluded from the curves of Fig. 7.6.

* For more recent discussions of this subject the reader is referred to Slichter,[133] Van Orstrand,[154] and Holmes,[56] all with good bibliographies. See also Lowan,[89] Bullard,[21] Jeffreys,[70] and Joly.[72]

contact? How long will it be before a point 2 m inside the surface will fall in temperature to 95°C? Assume for water, $k = 0.00143$, $\alpha = 0.00143$ cgs.

Ans. 90.2°C; 4.4 days

4. In the preceding problem calculate at what rate heat is passing out through each square meter of the boundary surface after 10 min.

Ans. 699 cal/sec

5. A 3,000-lb motor car traveling 30 mph is stopped in 5 sec by four brakes with brake bands of area 40 in.^{2} each, pressing against steel ($k = 26$, $\alpha = 0.48$ fph) drums, each of the above area. Assuming that the brake lining and drum surfaces are at the same temperature and that the heat is dissipated by flowing through the surface of the drums (assumed very thick), what maximum temperature rise might be expected?

SUGGESTION: Assume that this energy is converted into heat at a uniform rate and that this heat flows into the drum from the surface at a rate given by (7.16*c*). Compute the surface temperature T_s for the largest value of t, *i.e.*, 5 sec.

Ans. 132°F*

6. Show by a method of reasoning similar to that of Sec. 7.12, that if the plane surface of the solid is made impervious to heat, instead of being kept at constant temperature, then

$$T = \frac{\eta}{\sqrt{\pi}} \int_0^\infty f(\lambda)\, (e^{-(\lambda-x)^2\eta^2} + e^{-(\lambda+x)^2\eta^2})\, d\lambda$$

7. Water pipes are buried 1 m below the surface in concrete masonry ($\alpha = 0.0058$ cgs), the whole being at 8°C. If the surface temperature is lowered to -20°C, how long will it be before the pipes are in danger of freezing?

Ans. 9 days

8. If the initial temperature of the earth was 3900°C. throughout and it has been cooling 100 million years since then, with the surface at zero, plot its present state of temperatures as a function of the distance below the surface. (Use Kelvin's constants; *i.e.*, $\alpha = 0.0118$ cgs and $k = 0.0042$.)

9. Under the conditions of the previous problem compute the present loss of heat per square centimeter of surface per year. How thick a layer of ice would this melt?

Ans. 47.8 cal; 0.65 cm

10. In some modern heating installations the heat is supplied by pipes in the floor, *e.g.*, in a concrete slab on the ground. Assuming that such floor is in intimate contact (no insulation) with soil ($k = 0.5$, $\alpha = 0.015$ fph) initially at a uniform temperature 20°F lower than that of the pipe, calculate (Sec. 7.16) the rate of heat loss to the ground per square foot of floor area 100 hr and also 10,000 hr after the start of heating. Also, calculate the total loss at the end of these times. A large enough floor area to ensure linear flow is assumed.

Ans. 4.61 and 0.461 Btu/hr; 921 and 9210 Btu

*This is obviously too high since our calculation assumes this temperature throughout the 5 sec. A somewhat better treatment is indicated in Problem 7 of Sec. 8.14.

CHAPTER 8

LINEAR FLOW OF HEAT, II

In this chapter we shall continue the discussion of one-dimensional heat flow, taking up first the important matter of heat sources and following this with a treatment of the slab or plate and the radiating rod.

CASE III. HEAT SOURCES

8.1. We shall now make use of the conception of a *heat source*, an idea that has been used very successfully by Lord Kelvin[146]* and other writers in handling problems in heat flow. If a certain amount of heat is suddenly developed in each unit of area of a plane surface in a body, this surface becomes an *instantaneous source* of heat, while if the heat is developed continuously instead of suddenly, it is known as a *continuous source* or *permanent source*.†

8.2. Let Q units of heat be suddenly generated on each unit area of a plane in an infinite body, or on each unit area in some cross section of a long rod whose surface is impervious to heat. If the material is of specific heat c and density ρ, the unit of heat will raise the unit volume of the material $1/c\rho$ degrees. The quantity

$$S \equiv \frac{Q}{c\rho} \tag{a}$$

is called the *strength* of this instantaneous source. If Q' units are produced in each unit of time, then $S' \equiv Q'/c\rho$ is the strength of the permanent source.

8.3. Plane Source. Regard the plane $x = \lambda$ over which the instantaneous source of heat is spread as of thickness $\Delta\lambda$; then its

* "Mathematical and Physical Papers," II, p. 41 *ff.*

† The problem of Sec. 7.27 involved a special case of permanent sources with a volume distribution.

temperature when the heat is suddenly generated will be raised by

$$\frac{Q}{c\rho\Delta\lambda} = \frac{S}{\Delta\lambda} \text{ degrees} \tag{a}$$

and we have a case to be handled by (7.3*d*). The temperature at point x will be given by

$$T = \frac{S\eta}{\Delta\lambda\sqrt{\pi}} \int_{\lambda}^{\lambda+\Delta\lambda} e^{-(\lambda-x)^2\eta^2}\, d\lambda \tag{b}$$

since $f(\lambda) = 0$ outside these limits of integration. Now let the mean value of $e^{-(\lambda-x)^2\eta^2}$ between the above limits be $e^{-(\lambda'-x)^2\eta^2}$ where $\lambda < \lambda' < (\lambda + \Delta\lambda)$. Then,

$$T = \frac{S\eta}{\sqrt{\pi}} e^{-(\lambda'-x)^2\eta^2} \tag{c}$$

which, as $\Delta\lambda \to 0$, approaches the limit

$$T = \frac{S\eta}{\sqrt{\pi}} e^{-(\lambda-x)^2\eta^2} \tag{d}$$

where the heat source is at a plane λ distant from the origin. Shifting this to the origin, (*d*) becomes

$$T = \frac{S\eta}{\sqrt{\pi}} e^{-x^2\eta^2} \tag{e}$$

If we have a permanent source of constant strength S' located in a plane distant λ from the origin, which begins to liberate heat in a body initially at zero at time $t = 0$, we have at any time t later the summation of each effect $S = S'\,d\tau$ that acted at a time $t - \tau$ previously, τ being the time variable with limits 0 and t. Then, from (*d*)

$$T = \frac{S'}{2\sqrt{\pi\alpha}} \int_0^t e^{\frac{-(\lambda-x)^2}{4\alpha(t-\tau)}} (t-\tau)^{-\frac{1}{2}}\, d\tau \tag{f}$$

If the permanent source is at the origin, the expression is

$$T = \frac{S'}{2\sqrt{\pi\alpha}} \int_0^t e^{\frac{-x^2}{4\alpha(t-\tau)}} (t-\tau)^{-\frac{1}{2}}\, d\tau \tag{g}$$

Putting $\beta \equiv x/2\sqrt{\alpha(t-\tau)}$, this becomes, for positive values of x,

$$T = \frac{S'x}{2\alpha\sqrt{\pi}} \int_{\frac{x}{2\sqrt{\alpha t}}}^{\infty} \frac{e^{-\beta^2}}{\beta^2}\,d\beta = \frac{Q'x}{2k\sqrt{\pi}} \int_{x\eta}^{\infty} \frac{e^{-\beta^2}}{\beta^2}\,d\beta \qquad (h)$$

For the evaluation of this integral see Appendix B. See also (9.12d) and (9.12e). For negative values of x the upper limit is $-\infty$, giving the same value of T as for positive x.

8.4. Equation (8.3e) gives us temperatures at any point for any time if we have a linear flow of heat from an instantaneous source of strength S at the origin, the temperature of all other parts being initially zero. It is well to test the correctness of this solution by seeing if we can derive from it what is an inevitable conclusion from the conditions given, *viz.*, that the total amount of heat in the material at any time is just equal to the original amount Q (per unit area of section). From (8.3e) the quantity of heat in any element dx is

$$Tc\rho\,dx = \frac{Q\eta}{\sqrt{\pi}} e^{-x^2\eta^2}\,dx \qquad (a)$$

whence the total amount present in the body at any time is represented by

$$\int_{-\infty}^{\infty} Tc\rho\,dx = \frac{Q\eta}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2\eta^2}\,dx \qquad (b)$$

$$= \frac{Q\eta}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-\gamma^2}}{\eta}\,d\gamma = Q \qquad (c)$$

Since the additive effect of any number of such sources could be obtained by a summation of such terms as (8.3d), the formula (7.3d) may be regarded as applying to the case in which we start with an instantaneous source of strength $f(\lambda)\,d\lambda$ in each element of length $d\lambda$ of the solid or bar in the x direction.

8.5. Since it appears on expanding (8.3e) in a series that $T = 0$ ($x \neq 0$) when $t = 0$ and also when $t = \infty$, it must have a maximum value at some time t_1. To get this, differentiate (8.3e) and equate to zero,

$$\frac{\partial T}{\partial t} = \frac{S\eta}{\sqrt{\pi}} e^{-x^2\eta^2}\left(\frac{x^2}{4\alpha t^2} - \frac{1}{2t}\right) = 0 \qquad (a)$$

from which

$$t_1 = \frac{x^2}{2\alpha} \qquad (b)$$

Putting this value of t in (8.3e), we get for the value of this maximum

$$T_1 = \frac{S}{x\sqrt{2\pi e}} \tag{c}$$

8.6. Use of Doublets. Semiinfinite Solid, Initially at Zero, with Plane Face at Temperature $F(t)$. We shall now solve, with the aid of the concept of heat sources, an important problem in linear flow. This is the case of the semiinfinite solid initially at zero, whose boundary plane surface, instead of being at a constant temperature as in Sec. 7.14, is now a function of time.

We must find a solution of the conduction equation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \tag{7.1a}$$

subject to the conditions

$$T = 0 \qquad \text{when } t = 0 \tag{a}$$

and

$$T = F(t) \qquad \text{at} \quad x = 0 \tag{b}$$

We shall solve this problem by the use of a concept known as a "doublet." If a source and sink (negative source) of equal strength S are made to approach each other, while keeping constant the product of S and the distance $2b$ between them, this combination, in the limit, is called a doublet of strength $S_d \equiv 2bS$. With the aid of (8.3d) we may write at once the expression for the temperature at any point x due to an instantaneous doublet placed at the origin, *i.e.*, with the two sources at distance b on each side. This is

$$T = \frac{S}{2\sqrt{\pi\alpha t}}\left(e^{\frac{-(b-x)^2}{4\alpha t}} - e^{\frac{-(b+x)^2}{4\alpha t}}\right) \tag{c}$$

$$= \frac{S_d}{4b\sqrt{\pi\alpha t}}\, e^{\frac{-(b^2+x^2)}{4\alpha t}}\left(e^{\frac{bx}{2\alpha t}} - e^{\frac{-bx}{2\alpha t}}\right) \tag{d}$$

Expanding $e^{bx/2\alpha t}$ and $e^{-bx/2\alpha t}$ in a series (Appendix K) and dividing by b, we find at once that the term in parentheses, divided by b, becomes $x/\alpha t$ as b approaches zero. Then,

$$T = \frac{S_d x}{4\sqrt{\pi\alpha^3 t^3}}\, e^{-x^2/4\alpha t} \tag{e}$$

For a permanent doublet of constant strength S'_d located at the origin, with its axis in the x direction, we have the summation of the effects of each doublet element $S'_d\,d\tau$ that acted at a time $t - \tau$ previously, τ being the time variable (limits 0 and t) and t the time since the doublet was started. In this case we have

$$T = \frac{S'_d x}{4\sqrt{\pi\alpha^3}} \int_0^t e^{\frac{-x^2}{4\alpha(t-\tau)}} (t - \tau)^{-3/2}\,d\tau \tag{f}$$

For a permanent doublet of variable strength $\psi(t)$ this becomes

$$T = \frac{x}{4\sqrt{\pi\alpha^3}} \int_0^t \psi(\tau) e^{\frac{-x^2}{4\alpha(t-\tau)}} (t - \tau)^{-3/2}\,d\tau \tag{g}$$

which becomes, on writing

$$\beta \equiv \frac{x}{2\sqrt{\alpha(t - \tau)}}, \qquad i.e., \tau = t - \frac{x^2}{4\alpha\beta^2} \tag{h}$$

$$T = \frac{1}{\alpha\sqrt{\pi}} \int_{x\eta}^{\infty} \psi\left(t - \frac{x^2}{4\alpha\beta^2}\right) e^{-\beta^2}\,d\beta \tag{i}$$

This expression holds for positive values of x; for negative values the upper limit should be $-\infty$.

Now if we suppose a permanent doublet of strength $\psi = 2\alpha F(t)$ placed at the origin, we have

$$T = \frac{2}{\sqrt{\pi}} \int_{x\eta}^{\infty} F\left(t - \frac{x^2}{4\alpha\beta^2}\right) e^{-\beta^2}\,d\beta* \tag{j}$$

We have in (j) an expression that, from the manner of its formation, must be a solution of (7.1a)—a fact that can also be readily proved by direct differentiation. It also satisfies boundary conditions (a) and (b) and hence is the solution of our problem. It is to be noted that we are here interested only in positive values of x. If $F(t) = T_s$, a constant, (j) reduces at once to (7.14b) as it should.

If the initial temperature of the semiinfinite solid is $f(x)$ instead of zero, the solution may be obtained† by adding to (j) the equation (7.12g), the solution for the case of initial temperature $f(x)$ with boundary at zero.

* See Carslaw[27, pp. 17, 46] for a treatment of this problem by Duhamel's theorem.

† Carslaw and Jaeger.[27a, p. 46]

APPLICATIONS

8.7. Electric Welding. Two round iron

$$(k = 0.15, \qquad c = 0.105, \qquad \rho = 7.85 \text{ cgs})$$

bars 8 cm (3.1 in.) in diameter are being electrically welded end to end. If a current of 30,000 amp at 4 volts is required for 4 sec and if this energy is supposed to be all developed at the plane of contact, how far from the end will the temperature of 1200°C (2192°F) penetrate, if the initial temperature of the bars is taken to be 0°C?

The total heat developed will be

$$30{,}000 \times 4 \times 4 \text{ joules} = \frac{480{,}000}{4.18} \text{ cal, or } \frac{480{,}000}{4.18 \times 16\pi} \text{ cal/cm}^2 \qquad (a)$$

i.e., from (8.2*a*),

$$S = 2760 \text{ cgs} \qquad (b)$$

Hence, we have, from (8.5*c*),

$$1200 = \frac{S}{x\sqrt{2\pi e}} = \frac{2760}{x \cdot 4.13} \qquad (c)$$

or $x = 0.56$ cm; *i.e.*, the temperature of 1200°C will penetrate to a depth not greater than 0.56 cm (0.22 in.)—somewhat less, in fact, since the generation of heat is not instantaneous as the solution assumes.

8.8. Casting. A large flat plate of ferrous metal (use $k = 22$, $c = 0.15$, $\rho = 480$, heat of fusion = 90 fph) 1 in. (0.083 ft) thick is being cast in a sand ($k = 0.25$, $c = 0.24$, $\rho = 105$, $\alpha = 0.010$ fph) mold. Assuming that the pouring temperature is 2800°F while the mold is at 80°F, what will be the maximum temperature rise in the mold 6 in. from the plate, and when will this occur?

Because of the relatively high conductivity of the plate we can neglect its thickness and consider it a plane source. Then,

$$Q = 0.083 \times 480 \times 0.15 \times 2720 + 0.083 \times 480 \times 90 = 19{,}830 \text{ Btu/ft}^2 \qquad (a)$$

This gives a source in the sand of strength

$$S = \frac{19{,}830}{0.24 \times 105} = 787 \text{ fph} \qquad (b)$$

Then from (8.5*c*)

$$T_1 = \frac{787}{0.5 \times 4.13} = 381°\text{F temperature rise} \qquad (c)$$

giving a temperature in the sand of 461°F. From (8.5*b*) this will occur at

$$t_1 = \frac{0.25}{2 \times 0.01} = 12.5 \text{ hr} \qquad (d)$$

For half this distance away from the plate the temperature rise would be twice as much and the corresponding time a quarter as large as before.

The solution of the problem of Sec. 8.7 gives an idea of how far from the welded joint one might expect to find the grain of the material altered by overheating. From the second we could draw some conclusion as to how near such a casting, wood, say, might be safely located in the mold.

8.9. Temperatures in Decomposing Granite. We shall now take up a problem involving permanent sources with a volume distribution. While of some interest from the geological standpoint, it is difficult, and the solution of only one or two particular cases will be attempted.*

It has been noted in some instances that areas of granite undergoing decomposition are several degrees warmer than the surrounding rock. It is known that granite gives out heat during decomposition, the total amount being of the order of 100 cal/gm, but it is an extremely slow process, and our problem is to see if any reasonable assumption of the rate at which such heat is given off would serve to explain this increased temperature.

8.10. To be able to treat the case as a specific problem we shall assume first that the decomposing granite is in the form of a wall of thickness l, whose faces are kept at zero. Then if q_v cal/(sec)(cm^3) of the decomposing material are generated, we have for our fundamental equation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + \frac{q_v}{c\rho} \qquad (a)$$

* Attention is called to the "step method" (Secs. 11.16 to 11.22) for the approximate solution of problems like this, or even more complicated ones, by very simple mathematics.

with boundary conditions

$$T = 0 \qquad \text{at } x = 0 \text{ and } x = l \tag{b}$$

and

$$T = 0 \qquad \text{when } t = 0 \tag{c}$$

Let

$$u \equiv T + \Psi(x) \tag{d}$$

where $\Psi(x)$ is a function of x (only), yet to be determined. Replacing T by $u - \Psi(x)$ in (a),

$$\frac{\partial u}{\partial t} - \alpha\left[\frac{\partial^2 u}{\partial x^2} - \Psi''(x)\right] = \frac{q_v}{c\rho} \equiv B \tag{e}$$

But if we determine $\Psi(x)$ so that

$$\Psi''(x) = \frac{B}{\alpha} \tag{f}$$

or

$$\Psi(x) = \frac{Bx^2}{2\alpha} + bx + d \tag{g}$$

then,

$$\frac{\partial u}{\partial t} = \alpha\frac{\partial^2 u}{\partial x^2} \tag{h}$$

To satisfy (b) and also make $u = 0$ at $x = 0$ and $x = l$, $\Psi(x)$ must vanish at $x = 0$ and $x = l$; therefore,

$$d = 0 \quad \text{and} \quad b = -\frac{Bl}{2\alpha} \tag{i}$$

Then

$$\Psi(x) = \frac{B}{2\alpha}(x^2 - lx) \tag{j}$$

and

$$u = T + \frac{B}{2\alpha}(x^2 - lx) \tag{k}$$

or

$$T = u + \frac{B}{2\alpha}(lx - x^2) \tag{l}$$

The solution of the problem is then merely a question of determining u under the following conditions:

Fundamental equation,

$$\frac{\partial u}{\partial t} = \alpha\frac{\partial^2 u}{\partial x^2} \tag{m}$$

Boundary conditions,

$$u = 0 \qquad \text{at } x = 0 \text{ and } x = l$$

$$u = f(x) = \frac{B}{2\alpha}(x^2 - lx) \qquad \text{when } t = 0 \tag{n}$$

This is nothing but the problem of the slab with faces at zero,

which will be treated in Case IV, next to be considered. While in this particular example the form of $f(x)$ makes the determination of u a rather lengthy process, it offers no special difficulties and gives us as a final solution of the problem

$$T = \frac{q_v}{2k}\left(lx - x^2 - \frac{8l^2}{\pi^3}\sum_{m=2p+1}^{m=\infty}\frac{1}{m^3}e^{-\frac{\alpha m^2\pi^2 t}{l^2}}\sin\frac{m\pi x}{l}\right) \tag{o}$$

The curve of Fig. 8.1 has been computed with the use of the equation above, the rate q_v of heat generation being chosen so

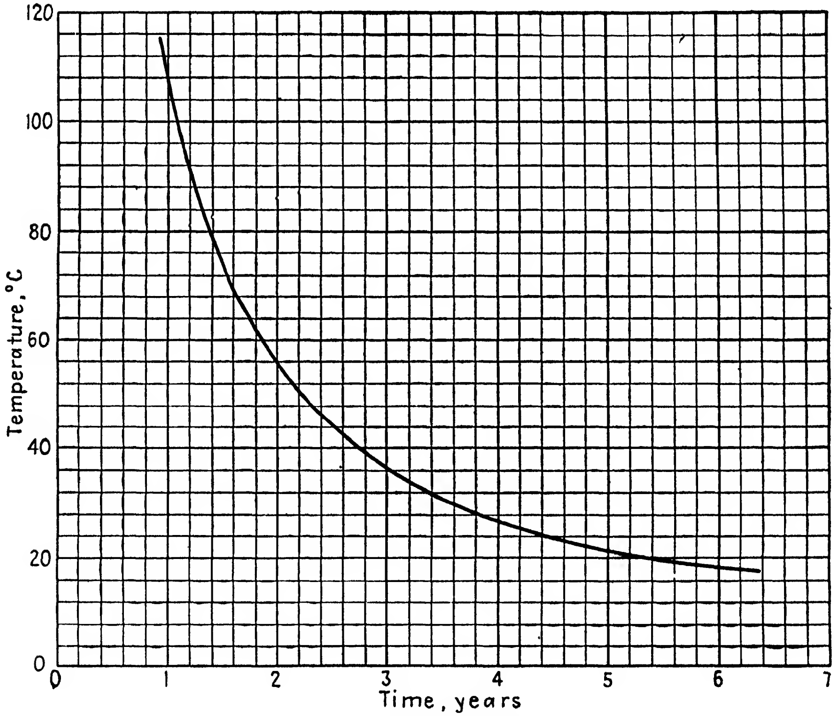

FIG. 8.1. Curve showing the relation between the final temperature in the center of a granite layer or wall 915 cm (30 ft) thick and the total time necessary to effect its decomposition, computed for the conditions of Sec. 8.10.

that the entire process of decomposition with the resultant generation of 100 cal/gm takes place in n years. The thickness of granite is taken as 915 cm (30 ft), and the time chosen as that for the completion of the process. The diffusivity is taken as 0.0155 cgs.

8.11. A second hypothetical case, much simpler than the above, is as follows: Suppose that this wall or slab of decomposing granite l cm thick is in contact on each side with ordinary granite. Suppose also that this slab is initially heated to some temperature T_0 about 50°C above that of the surrounding rock and allowed to cool for a year. This gives a temperature at the center, as may be readily computed from (7.4*b*), of $0.355T_0$, or about 17.7°C above that of the surrounding rock at some distance away. Now by differentiation of (7.4*b*) with respect to x and multiplication by 0.0081, the conductivity used here for granite, we get the rate of heat flow out through each face of this slab as

$$\frac{kT_0\eta}{\sqrt{\pi}}\,(1 - e^{-l^2\eta^2}) = 0.000057 \text{ cal/(cm}^2\text{)(sec)} \qquad (a)$$

for $l = 915$ cm.

So far we have taken no account of the heat of decomposition, for the above discussion is merely to find a reasonable assumption for the temperature distribution in this slab and the surrounding rock as we find it at present. We may now question at what rate decomposition would have to take place in order to furnish heat at just the rate required to maintain this temperature state steady for some time, and at once compute this rate as such that the 100 cal would be liberated, *i.e.*, the process finished, in about sixty-eight years.

The preceding discussion should enable the geologist to form some idea of the temperatures that might be caused by or explained by decomposition. Since the rate of such decomposition is generally supposed to be very much slower than that taken above, it is evident that a large thickness of such decomposing granite would be required to cause even a few degrees of excess temperature.*

8.12. Effect of Ground-temperature Fluctuations; Cold Waves. Equation (8.6*j*) enables a more accurate calculation of the effect of surface temperature fluctuations than is possible on the assumption that they are simple sine variations as was done in Sec. 5.10. As an example, suppose that a period of

* See Van Orstrand[153] in a discussion of a somewhat similar problem.

uniform ground temperature, say 0°C, is broken by a 3-day cold snap that causes a soil surface temperature of −12°C for this period, followed by a quick rise to the original 0°C. What is the temperature at a depth of 80 cm 5 days after the beginning of the cold snap? Assume $\alpha = 0.006$ cgs and neglect any latent-heat considerations.

Using (8.6j), put $t = 432{,}000$ sec and $x = 80$ cm. Note that $\tau[= t - (x^2/4\alpha\beta^2)]$ is the time variable and that

$$F\left(t - \frac{x^2}{4\alpha\beta^2}\right) = 0$$

save in the interval between $\tau = 0$ and $\tau = 259{,}200$ sec when it has the value −12°C. For $\tau = 0$ we have

$$4.32 \times 10^5 - \frac{6{,}400}{0.024\beta^2} = 0$$

which gives $\beta = 0.786$. Similarly, for $\tau = 259{,}200$ sec,

$$\beta = 1.24$$

Our solution then is

$$T = -12 \times \frac{2}{\sqrt{\pi}} \int_{0.786}^{1.24} e^{-\beta^2}\, d\beta = -2.24°\mathrm{C}$$

For cold or warm waves that are more complicated functions of time the solution is most readily arrived at by using a block curve for this function and evaluating the integral for the various limits involved.

Note that for any value of the time less than 3 days in the preceding problem the formula gives the same results as (7.14c), as it should.

8.13. Postglacial Time Calculations. A question of considerable interest to geologists is the matter of time that has elapsed since the last glacial sheet withdrew from any region. Calculations of such have been carried out by Hotchkiss and Ingersoll[57] with the aid of a series of carefully made temperature measurements in the deep Calumet and Hecla copper mines at Calumet, Mich.

Just as cold or hot waves produce an effect, though very limited in depth, on subsurface temperatures, so the retreat

of the ice sheet many thousands of years ago was followed by a warming of the surface that has produced a slight change in the geothermal curve of temperature plotted against depth. This change extends to thousands of feet below the surface. The problem then is to calculate from the magnitude of this change for various depths the time when the ice left and also the general surface temperature changes that have taken place since this time, *i.e.*, the thermal history of the region.

It is assumed that the last ice sheet lasted so long that the geothermal curve at its conclusion was a straight line and that the surface temperature was the freezing point of water. We shall show later how its slope is deduced. The present geothermal curve was determined by temperature measurements made with special thermometers and under special conditions at various depths reaching to nearly 6,000 ft below the surface. It was necessary, in order to secure virgin-rock temperatures unaffected by mining operations, to make measurements in special drill holes run many feet deep into the sides of newly made tunnels or "drifts" in which the rock surface had been exposed for only a few days. The curve as finally obtained is shown in the solid line of Fig. 8.2. The dashed line is the assumed geothermal curve at the end of the ice age.

Equation (8.6j) as it stands will not fit the boundary conditions of this problem, which are

$$T = F(t) \qquad \text{at } x = 0 \qquad (a)$$

and

$$T = Cx \qquad \text{when } t = 0 \qquad (b)$$

x being the depth below the surface. However, the addition of a term Cx to (8.6j) gives the equation

$$T = Cx + \frac{2}{\sqrt{\pi}} \int_{x\eta}^{\infty} F\left(t - \frac{x^2}{4\alpha\beta^2}\right) e^{-\beta^2}\, d\beta \qquad (c)$$

which is readily seen to satisfy the conduction equation (7.1a) quite as well as (8.6j) and also the conditions (a) and (b). The problem will be solved, then, when the form of $F(t)$ is determined, which, when inserted in (c), gives the best approximation to the present form of the geothermal curve.

It is obvious that it is much simpler to evaluate the integral

in (c) if the $F\left(t - \frac{x^2}{4\alpha\beta^2}\right)$ is taken as a constant between certain limits. This merely means the use of a block curve instead of a smooth curve. For example, if it is assumed that the glacial age ended 24,000 years ago and that the average surface tem-

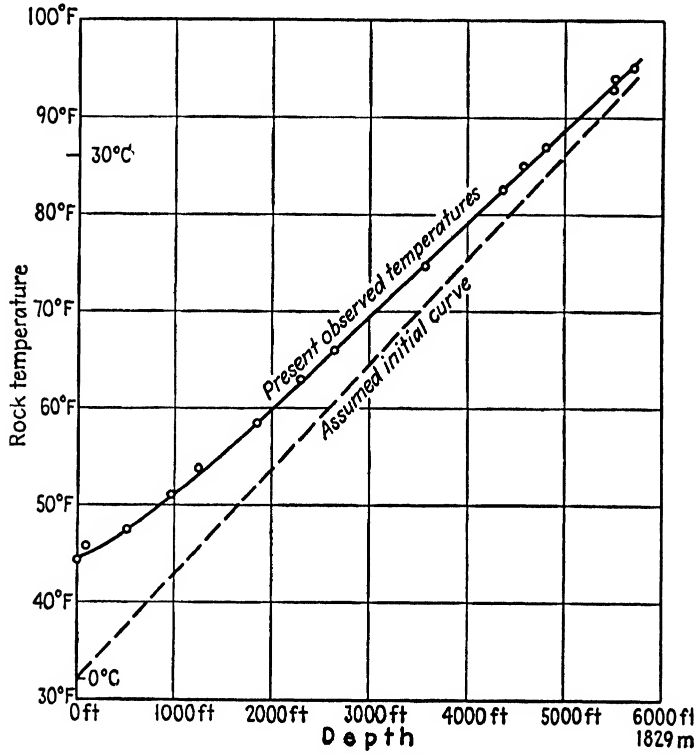

FIG. 8.2. Calumet and Hecla geothermal curves.

perature was 8°C for 18,000 years, followed by 6.83°C (its present value at this location) for the remaining 6,000 years to the present time, (c) would read

$$T = Cx + \frac{2}{\sqrt{\pi}}\left(8\int_{\frac{x}{2\sqrt{24{,}000n\alpha}}}^{\frac{x}{2\sqrt{6{,}000n\alpha}}} e^{-\beta^2}\,d\beta + 6.83\int_{\frac{x}{2\sqrt{6{,}000n\alpha}}}^{\infty} e^{-\beta^2}\,d\beta\right) \quad (d)$$

where n is the number of seconds in a year.

After α had been determined for two samples of the rock by

the method of Sec. 12.6, nearly fifty assumed thermal histories were tested by calculating values of T for each 500 ft (152 m) in depth, using equations of the type of (d). The constant C or slope of the assumed initial geothermal curve was determined by substituting the observed value of T at 5,500 ft (1,676 m) depth. This automatically makes the computed and observed

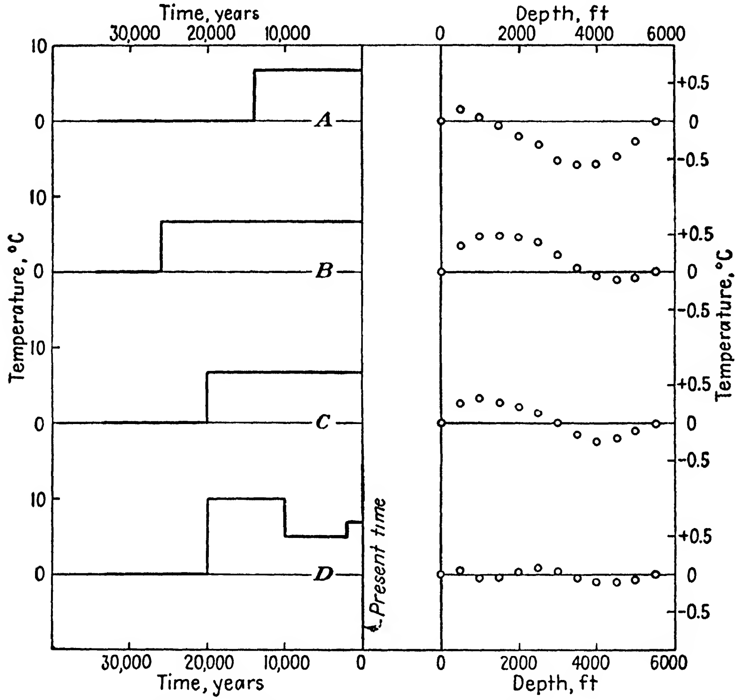

FIG. 8.3. Four assumed thermal histories and resultant deviations from the observed geothermal curve. Rock diffusivity taken as 0.0075 cgs.

value of T agree for the 5,500-ft point, and they must also agree at the surface, for one would naturally use 6.83°C, the present observed surface value, for the last part, at any rate, of the thermal history. There will be slight but entirely inconsequential variations in C, dependent on the thermal history used.

Four sample thermal histories are shown graphically in Fig. 8.3, as well as the resultant deviations from the observed geothermal curve. These are the differences between the values of T calculated by an equation of the type of (d) for each thermal history, and the observed values. In history A the Wisconsin

ice sheet was supposed to melt away from this region some 14,000 years ago with the present average surface temperature of 6.83°C dating from that time. In B the date was 26,000 years ago, and in C 20,000 years. In D the assumption is that the ice ended 20,000 years ago and for 10,000 years the surface averaged 10°C in temperature, *i.e.*, the climate was somewhat warmer than at present. This was followed by 8,000 years at 5°C, and then for 2,000 years to the present time the temperature was 6.83°C. This value of $F(t)$ gave about the smallest deviations of any tested and accordingly represents the best conclusions one can draw from this work.

8.14. Problems

1. Derive (7.3*d*) and (7.12*d*) on the basis of heat sources (see Sec. 8.4).

2. In electrically welding two large iron ($k = 0.15$, $c = 0.105$, $\rho = 7.85$ cgs) bars 2640 cal is suddenly developed in each square centimeter of contact plane. If the initial temperature is 30°C, when will the maximum occur at 15 cm from this plane and what will be its value? *Ans.* 618 sec; 81.7°C

3. A plate of lead ($k = 0.083$, $c = 0.030$, $\rho = 11.3$, latent heat of fusion 6 cgs) is cast in a sand ($k = 0.0010$, $c = 0.25$, $\rho = 1.7$ cgs) mold. If the mold is initially at 25°C while the lead is poured at 400°C, what will be the maximum temperature 3 cm away and when will this occur? The plate is 1 cm thick. *Ans.* 62°C; 1,913 sec

4. Show from (8.6*i*) that, if we have a permanent doublet of strength $2\alpha T_s$ at the origin, we get at once the solution of the case treated in Sec. 7.14 [Equation (7.14*b*)].

5. Soil ($\alpha = 0.015$ fph) initially at 34°F has its surface chilled to 16°F for two days, after which the surface returns to its original temperature. What is the temperature 2 ft underground 3 days after the cold wave began? *Ans.* 31.2°F

6. A steel ($\alpha = 0.121$ cgs) rod at 0°C, whose sides are thermally insulated, has its end suddenly heated by an electric arc to 1400°C for 1 min and then chilled again to 0°C. What is the temperature 5 cm from the end 3 min after the heating was started? *Ans.* 133°C

7. Solve Problem 5 of Sec. 7.28 by the method of heat sources, using (8.3*f*) or (8.3*h*) and assuming that the heat is generated at a uniform rate over the 5 sec. (Note that, since these equations assume heat flow in both directions, we must use double the present rate of heat generation.) *Ans.* 84°F

CASE IV. SOLID WITH TWO PARALLEL BOUNDING PLANES—THE SLAB OR PLATE

8.15. In this case we have to deal with a body bounded by two parallel planes distant l apart, with the initial temperature

condition of the body given. The problem is to find the subsequent temperature for any point. The solution will of course fit equally well the case of a short rod with protected surface.

8.16. Both Faces at Zero. The boundary conditions here are

$$T = 0 \qquad \text{at } x = 0 \tag{a}$$
$$T = 0 \qquad \text{at } x = l \tag{b}$$
$$T = f(x) \qquad \text{when } t = 0 \tag{c}$$

Now we have already seen (Sec. 7.2) that

$$T = e^{-\alpha\gamma^2 t} \sin \gamma x \tag{d}$$

and

$$T = e^{-\alpha\gamma^2 t} \cos \gamma x \tag{e}$$

are particular solutions of the fundamental equation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \tag{7.1a}$$

Form (d) satisfies (a) for any value of γ, and also (b) if $\gamma = m\pi/l$ where m is a whole number. It does not, as it stands, fulfill (c), but it may be possible to combine a number of terms like (d) and secure an expression that will be a solution of (7.1a) and that satisfies (c). For

$$T = B_1 e^{\frac{-\pi^2\alpha t}{l^2}} \sin \frac{\pi x}{l} + B_2 e^{\frac{-4\pi^2\alpha t}{l^2}} \sin \frac{2\pi x}{l} + B_3 e^{\frac{-9\pi^2\alpha t}{l^2}} \sin \frac{3\pi x}{l} + \cdots \tag{f}$$

is still a solution of (7.1a), satisfying (a) and (b), which reduces, when $t = 0$, to

$$T = B_1 \sin \frac{\pi x}{l} + B_2 \sin \frac{2\pi x}{l} + B_3 \sin \frac{3\pi x}{l} + \cdots \tag{g}$$

and from Sec. 6.8 this equals $f(x)$ if the function fulfills the conditions of Sec. 6.1 between 0 and l, and if

$$B_m = \frac{2}{l} \int_0^l f(\lambda) \sin \frac{m\pi\lambda}{l} \, d\lambda \tag{h}$$

The solution of our problem then is

$$T = \frac{2}{l} \sum_{m=1}^{\infty} \left[e^{\frac{-m^2\pi^2\alpha t}{l^2}} \sin \frac{m\pi x}{l} \int_0^l f(\lambda) \sin \frac{m\pi\lambda}{l} \, d\lambda \right] \tag{i}$$

If $f(\lambda) = T_0$, a constant, and if the surfaces are at T_s, we may write from (i),

$$T - T_s =$$

$$(T_0 - T_s)\frac{2}{l}\sum_{m=1}^{\infty}\left[e^{\frac{-m^2\pi^2\alpha t}{l^2}}\frac{l}{m\pi}(1 - \cos m\pi)\sin\frac{m\pi x}{l}\right] \quad (j)$$

which holds for either heating or cooling. Only odd terms in m are present; so we have, for the middle of the slab,

$$\frac{T_c - T_s}{T_0 - T_s} = \frac{4}{\pi}\left(e^{\frac{-\pi^2\alpha t}{l^2}} - \frac{1}{3}e^{\frac{-9\pi^2\alpha t}{l^2}} + \frac{1}{5}e^{\frac{-25\pi^2\alpha t}{l^2}} - \cdots\right) \quad (k)$$

The series

$$S(z) \equiv \frac{4}{\pi}\left(e^{-\pi^2 z} - \frac{1}{3}e^{-9\pi^2 z} + \frac{1}{5}e^{-25\pi^2 z} - \cdots\right) \quad (l)$$

is evaluated in Appendix G (z obviously equals $\alpha t/l^2$). For a slab initially at zero, heated by surfaces at T_s, (k) becomes

$$T_c = T_s[1 - S(z)] \quad (m)$$

while, for cooling from an initial temperature T_0 with surfaces at zero, the equation is simply

$$T_c = T_0 S(z) \quad (n)$$

8.17. Adiabatic Cases—Slab with Nonconducting Faces. If the faces instead of being kept at constant temperature are impervious to heat, we shall have the same differential equation but quite different boundary conditions; *viz.*,

$$\frac{\partial T}{\partial x} = 0 \qquad \text{at } x = 0 \quad (a)$$

$$\frac{\partial T}{\partial x} = 0 \qquad \text{at } x = l \quad (b)$$

$$T = f(x) \qquad \text{when } t = 0 \quad (c)$$

Conditions (a) and (b) are fulfilled by solution (8.16e) if

$$\gamma = \frac{m\pi}{l}$$

just as before, and (c) may be satisfied by combining a number

of terms of this type. This gives

$$T = \frac{2}{l}\left\{\frac{1}{2}\int_0^l f(\lambda)\,d\lambda + \sum_{m=1}^{m=\infty}\left[e^{\frac{-\alpha m^2\pi^2 t}{l^2}}\cos\frac{m\pi x}{l}\int_0^l f(\lambda)\cos\frac{m\pi\lambda}{l}\,d\lambda\right]\right\} \qquad (d)$$

8.18. If only one face is nonconducting, the other being kept at zero, the solution is contained in equation (8.16*i*). This may be shown by the same considerations that were used in Sec. 7.6, *i.e.*, by imagining a nonconducting plane cutting through the center of a slab of double thickness, parallel to its faces, where the temperature conditions are supposed perfectly symmetrical on each side of such a plane. There would then be no tendency to a flow of heat across such a surface, and hence placing a nonconducting division plane there and removing half of the slab will not affect the solution in any way. Therefore, in handling a problem of this nature, *i.e.*, one face impervious to heat, we solve it as a case of a slab of *twice* the thickness, and the temperatures of the nonconducting face would be found as those at the *middle* of the slab of double thickness.

APPLICATIONS

8.19. The Theory of the Fireproof Wall. With the aid of the foregoing deductions we can now develop a theory that finds immediate application to a large number of practical problems, *viz.*, that of heat penetration into a slab or wall, one side of which is subjected to sudden heating, as by fire; or, as we shall call it for brevity, the "theory of the fireproof wall." It is to be understood that this theory applies only to the purely thermal aspects of the question of fire-protecting walls and floors and not at all to the very important considerations of strength, ability to withstand heating and quenching, and other questions that must be largely determined by experiment.

We shall treat the problem for four cases of somewhat differing conditions. It is assumed in all cases that the wall is relatively homogeneous in structure, a condition that would be fulfilled by practically all masonry or concrete walls, floors, or

chimneys. For hollow tiling or other cellular structure the theory would not apply directly but would still afford at least an indication of the laws for these cases. It is also assumed that the wall is initially at about the same temperature throughout its thickness, as would be true in almost every practical example. All temperatures are measured from the initial "zero" of the wall.

8.20. Case A. The conditions assumed for this case are that the front face of the wall is suddenly raised to the temperature T_s and maintained there, while the rear face is protected so that it suffers no loss of heat. It is desired to know the rise in temperature of the rear face for various intervals of time. The latter condition is fulfilled sufficiently well by a wall that is backed by wood, *i.e.*, door casing, or better by a concrete or masonry floor on which is piled poorly conducting (*e.g.*, combustible) material.

As explained in Sec. 8.18, such a case as this, involving an impervious surface, can be treated as that of a slab of twice the thickness, the rear (impervious) face of the wall corresponding to the middle of the slab ($x = \frac{1}{2}l$). Accordingly (8.16m) gives the expression for the rear face temperature, for a wall initially at zero, *i.e.*,

$$T = T_s[1 - S(z)] \tag{a}$$

where $z = \alpha t/l^2$. Note that l in this case is *twice* the wall thickness. Values of $S(z)$ are given in Appendix G.

8.21. Case B. This differs from the preceding in that the temperature of the front face is supposed to rise gradually instead of suddenly. If the rise is rapid at first, as it would be in most cases—*e.g.*, if the wall were exposed to a flame—an approximate solution may be arrived at by the device suggested in discussing the removal of shrunk-on fittings (Sec. 7.21), *i.e.*, the assumption of an added thickness whose outer surface is suddenly raised to, and kept at, a constant temperature T'_s. By properly choosing T'_s as well as the thickness to be added, a temperature-time curve can be found for the plane representing the original surface, nearly like many actual heating curves; the computation is then carried out accordingly. The results

obtained, however, are generally only slightly different from those for Case A if the mean value of T_s is used.

8.22. Case C. We have here an important difference to take account of in the conditions. While the front surface is supposed to be suddenly brought to the temperature T_s as in Case A, the rear surface in the present case is supposed to lose heat by radiation and convection instead of being protected, and hence will not rise to as high a temperature as in Case A.

The rigorous handling of this problem is extremely difficult and would be well beyond the limits of the present work, but, as in many previous cases, it is still possible to reach a solution accurate enough for all practical purposes, and at not too great an expense of labor. This may be done as follows: In the treatment of the semiinfinite solid with boundary at zero (Sec. 7.12) we found that the equations could be deduced from those for the infinite solid by a suitable assumption for the temperatures on the negative side of the origin, *i.e.*, for $f(-\lambda)$, the latter being so determined that the boundary should remain constantly at zero. Now if the boundary instead of being at zero radiates with an emissivity h, this condition can be introduced* by putting into the relation [identical with (7.3*d*)]

$$T = \frac{1}{2\sqrt{\pi\alpha t}} \int_0^\infty \left[f(\lambda) e^{\frac{-(\lambda-x)^2}{4\alpha t}} + f(-\lambda) e^{\frac{-(\lambda+x)^2}{4\alpha t}} \right] d\lambda \qquad (a)$$

the condition that

$$f(-\lambda) = f(\lambda) - 2\frac{h}{k} e^{-\frac{h}{k}\lambda} \int_0^\lambda f(\gamma) e^{\frac{h}{k}\gamma}\, d\gamma \qquad (b)$$

This gives the temperatures for a semiinfinite medium with radiating surface and initial temperature conditions determined for $f(\lambda)$. Now let us make the assumption that $f(\lambda)$ has the value zero for a distance b from the radiating face, and $2T_s$ from there to infinity. This gives the somewhat complicated equation

$$T = \frac{2T_s}{\sqrt{\pi}} \int_{(b-x)\eta}^{(b+x)\eta} e^{-\beta^2}\, d\beta + 2T_s e^{(b+x)\frac{h}{k} + \frac{h^2}{k^2}\alpha t} \left\{ 1 - \Phi\left[\left(b + x + 2\frac{h}{k}\alpha t \right) \eta \right] \right\} \qquad (c)$$

* See Weber-Riemann.[160, II, Art. 39]

and if we investigate with the aid of this equation the temperature in the plane distant b from the radiating face, we find that, for small values of h and not too small values of b, this is almost constant for a considerable time and has the value T_s.

We have, then, the solution of our problem in the above equation. This plane that is kept at T_s corresponds to the

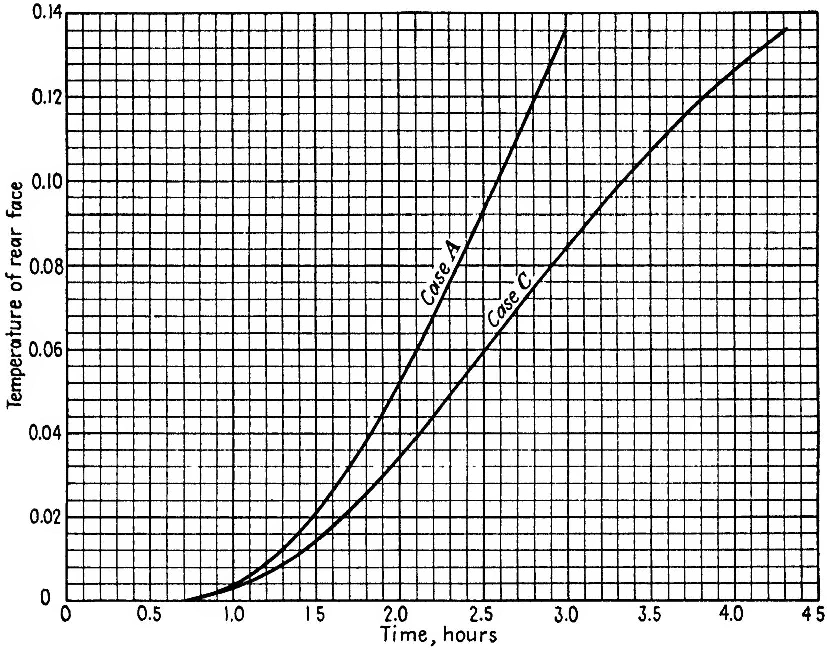

FIG. 8.4. Temperatures of the rear face of a concrete wall 20.3 cm (8 in.) thick, whose front face is heated to T_s; computed for the conditions of Cases A and C. Ordinates are fractions of T_s.

front face of the wall whose thickness is b, and the temperatures of the rear or radiating face will be given by putting $x = 0$ in this equation. The value of the constant h may be taken for small ranges of temperature at about 0.0003 cal/(sec)(cm^2)(°C) above the temperature of the surroundings, for an average surface such as a wall (see Appendix A). Strong convection such as a wind, or higher temperature differences, will increase this figure considerably; in some cases, however, it may be even less than the above value.

To gain some idea of the difference of the results for this case

and for Case A, a few computations have been carried out with (c) and plotted in Fig. (8.4). These are for a wall of concrete ($\alpha = 0.0058$ cgs) 20.3 cm (8 in.) thick, whose front face is heated to T_s. For 2 hr, under these conditions, the temperatures of the rear face for Case C are lower than they would be for Case A in the ratio of 35 to 53.

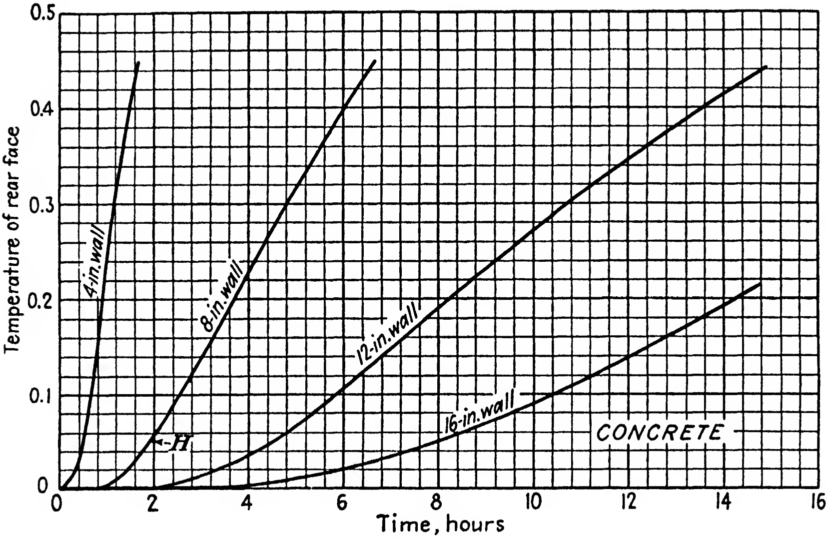

FIG. 8.5.—Computed curves showing the rise in temperature of the rear faces of walls of concrete ($\alpha = 0.0058$ cgs), whose front faces are suddenly heated to, and afterwards maintained at, T_s. See Secs. 8.24 and 8.25. Ordinates are fractions of T_s.

8.23. Case D. This differs from the last only in the supposition that the temperature rises gradually instead of suddenly. No attempt* will be made at treating this case mathematically, but from the conclusions reached for Case B we are reasonably safe in handling it as Case C, using a mean value for the temperature T_s.

8.24. Discussion of the General Principles. Having treated in detail the several cases, we may now draw some general conclusions in regard to thermal insulation under fire conditions. From the preceding discussion we see that Case A is the one from which we can most safely make these deductions; for B and

* For a fairly approximate treatment the method used for Case B might be followed; *i.e.*, the assumption of a small added thickness.

D are more or less minor modifications, while C would invariably lead to lower results. Hence, for a margin of safety we shall make our deductions largely from (the ideal) Case A.

The first conclusion to be drawn from (8.20*a*) is that the temperature of the rear face is a function of α rather than of k. In other words, the insulating value of material for such a wall is dependent not alone on its conductivity, but rather on its conductivity divided by the product of its specific heat and density,

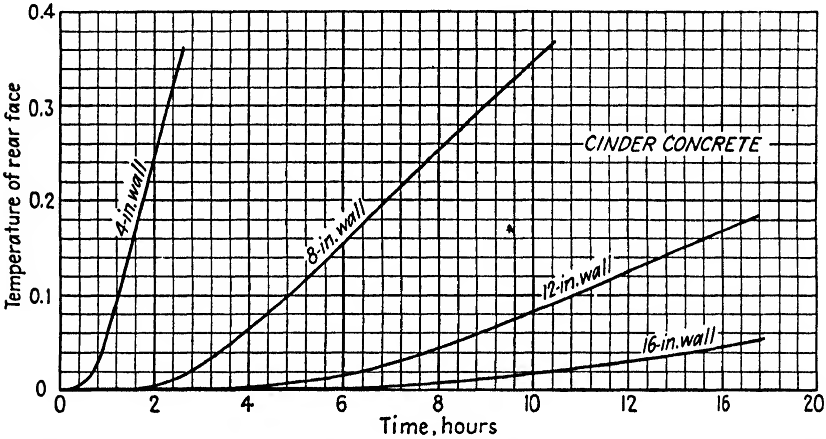

FIG. 8.6. Computed temperature-time curves for the rear faces of walls of cinder concrete (α = 0.0031 cgs). Ordinates are fractions of T_s.

i.e., its *diffusivity*. Material for such purpose should therefore have as low a conductivity and as high a density and specific heat as possible, for if the density happens to be low, it may prove no better insulator than something of higher conductivity but of correspondingly higher density.

The second conclusion from (8.20*a*) is that any change that alters t and l^2 in the same proportion does not affect the temperature T of the rear surface of the wall. In other words, for a given temperature rise of the rear face the time will vary as the square of the thickness. Since one measure of the effectiveness of such a fireproof wall or floor would be the time to which it would delay the penetration of a dangerously high temperature to the rear face, this makes the efficiency of such a wall or floor proportional to the square of its thickness (cf. the "law of times" in Sec. 7.15).

These conclusions are represented graphically in the curves of Figs. 8.5 to 8.7. The temperature T of the rear face of a wall whose front face is at T_s is expressed for various times and thicknesses of wall in fractions of T_s.

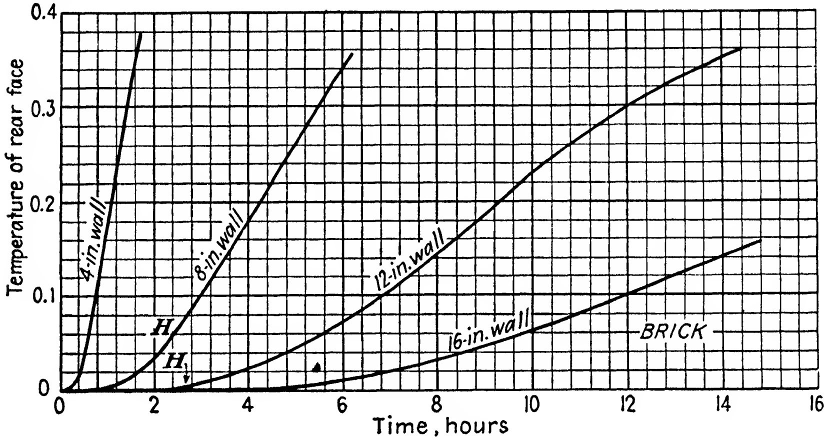

FIG. 8.7. Computed temperature-time curves for the rear faces of walls of building brick ($\alpha = 0.0050$ cgs). Ordinates are fractions of T_s.

8.25. Experimental. The following simple experimental check on the preceding conclusions was tried by the authors: A plate of hard unglazed porcelain 0.905 cm thick was heated on one surface by the sudden application of hot mercury and the temperature rise of the other surface, which was protected from loss of heat by loose cotton wrappings, was measured with a small thermoelement. The process was repeated for a similar plate of thickness 1.780 cm, the temperatures being plotted in Fig. 8.8. Since the diffusivity of the porcelain was not known, it was computed from the determination for the thinner plate that $T = \frac{1}{2}T_s$ at time 52 sec. This gives $\alpha = 0.0060$ cgs, and the two theoretical curves were computed from this value. Two plates of each thickness were tested, and it is to be noted that the agreement with the theoretical curve is at least as close as that between the two sets of observations. The whole is in reasonable agreement with the "law of times."

On a larger scale there are available the fire tests on various walls made by R. L. Humphrey.[60] These were 2-hr tests, mostly on 8-in. walls, the temperature T_s of the front faces

being in the neighborhood of 700°C. His results have been plotted, where possible, in the curves of Figs. 8.5 to 8.7 being denoted by the symbol H. The agreement, overlooking radiation losses, for the case of concrete is good.

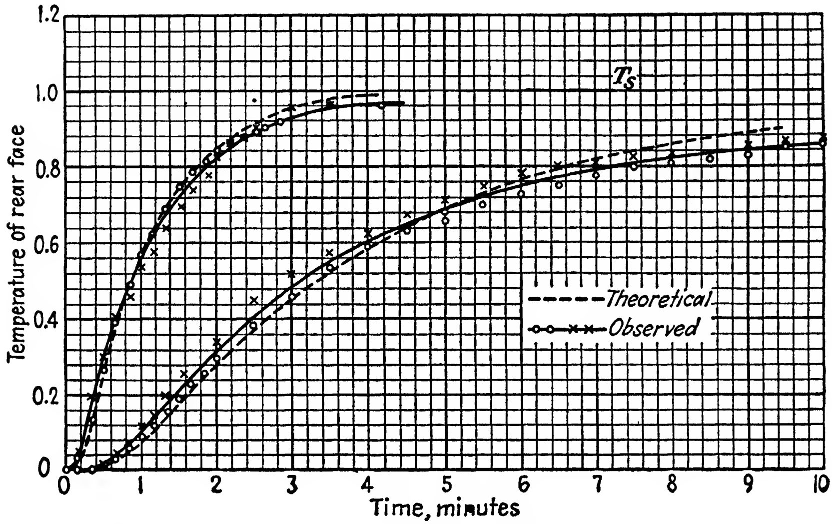

FIG. 8.8. Theoretical and observed temperature-time curves for the rear faces of miniature walls of porcelain ($\alpha = 0.0060$ cgs), initially at zero, the temperature of whose front faces was suddenly raised to T_s and maintained there during the experiment.

8.26. Molten-metal Container; Firebrick. We may make brief mention of a number of other problems to which the foregoing principles apply more or less directly. For example, take the case of a container lined with magnesia firebrick 30.5 cm thick, in which molten metal at an average temperature of 1300°C is kept for two or three hours. How hot may the outside of the brick be expected to get if the radiation from the surface is small? Using $\alpha = 0.0074$ cgs and $l = 61$ cm, we find, with the aid of (8.16m) that the temperature of the outside would be expected to rise only 8°C in 2 hr while in 4 hr it should not exceed 95°C.

In a number of practical cases it is desirable to know to what extent and how rapidly the temperature in the inside of a brick follows that of the outside. This is of particular interest in connection with the burning of brick and also in the case of

the "regenerator," where heat from flue gases is stored up in a checkerwork wall of firebrick, to be utilized shortly in heating other gases. Using $\alpha = 0.0074$ cgs, we find that the center of such a brick 6.35 cm (2.5 in.) thick—the larger dimensions being of little influence if the two flat sides are exposed (but see Sec. 9.44)—will rise in 5 min to 0.26 of the temperature of the faces, in 10 min to 0.57, and in 20 min to 0.85. For building brick of perhaps two-thirds this diffusivity the figures would be 0.12 for 5 min, 0.38 for 10 min, and 0.70 for 20 min.

8.27. Optical Mirrors. In the process of finishing huge telescopic mirrors it is necessary that they be allowed to remain in a constant-temperature room before testing, until the glass is at sensibly the same temperature throughout. For such a glass ($\alpha = 0.0057$ cgs) mirror 25 cm thick we can calculate from (8.16m) that if the surface temperature is changed by T_s the change at the center is 90 per cent of this after 7.8 hr. For 14.2 hr the figure would be 98.7 per cent.

8.28. Vulcanizing. The process of vulcanizing tires lends itself to some theoretical treatment along the preceding lines, in spite of the fact that the "slab" involved here, *i.e.*, the carcass of the tire, is sharply curved, with radius of only a few inches in some cases. We may question how long it would take for the central layer of a tire initially at 30°C to reach 120°C if the steam temperature in the forms on each side is 140°C. Assume a tire thickness of 16 mm and a diffusivity of 0.001 cgs. Then, from (8.16k) we have

$$\frac{120 - 140}{30 - 140} = S\left(\frac{0.001t}{2.56}\right)$$

Using Appendix G, we find $t = 506$ sec.

8.29. Fireproof Containers; Annealing Castings. While a large number of other applications of the foregoing theory might be mentioned, such as numerous cases of fireplace insulation, resistor-furnace insulation, fireproof-safe construction, and the like, we shall content ourselves with only one or two more examples.

The first is the matter of a fireproof container made with a thickness of 3 in. of special cement (use $\alpha = 0.012$ fph). If the front surface is raised to 500°F, how long would it be before

the inside surface, considered as adiabatic, would reach 300°F, assuming an initial temperature of 70°F? Using (8.16k), we have at once $(300 - 500)/(70 - 500) = S(\alpha t/l^2)$. From Appendix G we have $0.012t/0.5^2 = 0.102$, or $t = 2.1$ hr.

A second problem is that of annealing castings; *i.e.*, the question of how long the heating must continue to bring the interior to the desired temperature. We may readily compute that for a metal casting ($\alpha = 0.173$ cgs) in the form of a plate 30.5 cm or 1 ft in thickness it would take 23 min for the center to rise to within 90 per cent of the temperature of the faces, provided these were quickly raised to their final temperature. For a plate of half this thickness it would take only one-quarter the time. If the faces were gradually heated, the process would take longer, but the difference between the outside and inside temperatures would be lessened.

8.30. Problems

1. A plate of steel ($\alpha = 0.121$ cgs) of thickness 2.54 cm and temperature 0°C is to be tempered by immersion in a bath of stirred molten metal at T_s. How long should it be left to assure that the steel is throughout within 98 per cent of this higher temperature? *Ans.* 23 sec

2. A fireplace is insulated from wood by 15 cm of firebrick ($\alpha = 0.0074$ cgs). If the face is kept for some time at 425°C, how long will it be before the wood at the rear will char, supposing this to occur at 275°C? Initial temperature is 25°C. How long for a thickness of 25 cm? *Ans.* 4.2 hr; 11.6 hr

3. A 2-cm thick rubber ($\alpha = 0.001$ cgs) tire is to be vulcanized at 150°C, initial temperature being 20°C. How long will it be before the center will attain 145°C? *Ans.* 1,420 sec

4. Compare the results for the three following problems based on Cases I and II of Chap. 7 and Case IV of this chapter. A plate of copper ($k = 0.918$, $c = 0.0914$, $\rho = 8.88$, $\alpha = 1.133$ cgs) 10 cm thick and at T_0 is placed between two large slabs of similar material at zero; how long will it be before the center will fall in temperature to $\frac{1}{2}T_0$? If instead of a plate we have a large mass originally at T_0, while the surface is afterward kept at zero, how long will it be before the temperature 5 cm in from the surface will fall to $\frac{1}{2}T_0$? If the slab is of the same thickness as in the first case, but the faces are kept at zero, solve this problem for the center. *Ans.* 24.3 sec; 24.3 sec; 8.3 sec

5. A sheet of ice ($k = 0.0052$, $c = 0.502$, $\rho = 0.92$, $\alpha = 0.0112$ cgs) 5 cm thick, in which the temperature varies uniformly from zero on one face to -20°C on the other, has its faces protected by an impervious covering. What will be the temperature of each face after 10 min?

Ans. -10.56°C and -9.44°C

CASE V. LONG ROD WITH RADIATING SURFACE

8.31. This differs from Cases I and II of Chap. 7 in that there is a continual loss of heat by radiation from the surface of the rod. We have already handled the steady state for this case in Secs. 3.5 to 3.8, where we found that the Fourier equation had to be modified by the addition of a term taking account of the radiation and became

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} - b^2 T \qquad (a)$$

We shall assume as before that the rod is so thin that the temperature is sensibly uniform over the cross section, and that the surroundings are at zero.

8.32. Initial Temperature Distribution Given. We must seek a solution of (8.31a), subject to the conditions

$$T = f(x) \qquad \text{when } t = 0 \qquad (a)$$
$$T = 0 \qquad \text{when } t = \infty \qquad (b)$$

Now the substitution $\quad T = ue^{-b^2 t} \qquad (c)$

reduces (8.31a) at once to $\quad \dfrac{\partial u}{\partial t} = \alpha \dfrac{\partial^2 u}{\partial x^2} \qquad (d)$

where u fulfills the condition

$$u = f(x) \qquad \text{when } t = 0 \qquad (e)$$

and indirectly (b), since u is finite. But this is identical with Case I; thus, the solution for u is given by (7.3f). Using this, we may write at once

$$T = \frac{e^{-b^2 t}}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x + 2\beta \sqrt{\alpha t}) e^{-\beta^2} \, d\beta \qquad (f)$$

In other words, this differs from the nonradiating case only by the factor $e^{-b^2 t}$.

8.33. One End of Rod at Zero; Initial Temperature Distribution Given. The boundary conditions are

$$T = 0 \qquad \text{at } x = 0 \qquad (a)$$
$$T = f(x) \qquad \text{when } t = 0 \qquad (b)$$

If we make the substitution (8.32c), then u must satisfy (8.32d) and also the conditions

$$u = 0 \qquad \text{at } x = 0 \qquad (c)$$
$$u = f(x) \qquad \text{when } t = 0 \qquad (d)$$

Since this is the case already treated in Sec. 7.12, we may write, using (7.12g),

$$T = \frac{e^{-b^2t}}{\sqrt{\pi}}\left[\int_{-x\eta}^{\infty} f\left(\frac{\beta}{\eta} + x\right)e^{-\beta^2}\,d\beta - \int_{x\eta}^{\infty} f\left(\frac{\beta}{\eta} - x\right)e^{-\beta^2}\,d\beta\right] \qquad (e)$$

8.34. End of Rod at Constant Temperature T_s; Initial Temperature of Rod Zero. We cannot solve this problem directly, like the two preceding, as an extension of cases already worked out; for the boundary condition $T = T_s$ at $x = 0$ would mean $u = T_s e^{b^2t}$ at $x = 0$, which would not fit any case we have treated. But we can handle this case with (8.33e) by the aid of an ingenious device* whereby we first solve the problem for the boundary conditions

$$T = 0 \qquad \text{at } x = 0 \qquad (a)$$
$$T = -T_s e^{-bx/\sqrt{\alpha}} \qquad \text{when } t = 0 \qquad (b)$$

Applying (8.33e) to this case, we get, on simplifying,

$$T = \frac{T_s}{\sqrt{\pi}}\left(e^{\frac{bx}{\sqrt{\alpha}}}\int_{x\eta}^{\infty} e^{-(b\sqrt{t}+\beta)^2}\,d\beta - e^{\frac{-bx}{\sqrt{\alpha}}}\int_{-x\eta}^{\infty} e^{-(b\sqrt{t}+\beta)^2}\,d\beta\right) \qquad (c)$$

Now

$$T = T_s e^{-bx/\sqrt{\alpha}} \qquad (d)$$

is a particular solution of (8.31a), as is also (c) above. Thus, the sum of (c) and (d),

$$T = T_s\left(e^{\frac{-bx}{\sqrt{\alpha}}} + \frac{e^{\frac{bx}{\sqrt{\alpha}}}}{\sqrt{\pi}}\int_{x\eta}^{\infty} e^{-(b\sqrt{t}+\beta)^2}\,d\beta - \frac{e^{\frac{-bx}{\sqrt{\alpha}}}}{\sqrt{\pi}}\int_{-x\eta}^{\infty} e^{-(b\sqrt{t}+\beta)^2}\,d\beta\right) \qquad (e)$$

is still a solution of (8.31a), which, moreover, fits our present boundary conditions, *viz.*,

$$T = T_s \qquad \text{at } x = 0 \qquad (f)$$
$$T = 0 \qquad \text{when } t = 0 \qquad (g)$$

* Cf. Byerly.[23, p. 90]

We may simplify this somewhat by writing

$$\gamma \equiv b\sqrt{t} + \beta \tag{h}$$

and hence

$$d\gamma = d\beta \tag{i}$$

in (e). This gives

$$T = T_s\left(e^{\frac{-bx}{\sqrt{\alpha}}} + \frac{e^{\frac{bx}{\sqrt{\alpha}}}}{\sqrt{\pi}}\int_{b\sqrt{t}+x\eta}^{\infty} e^{-\gamma^2}\,d\gamma - \frac{e^{\frac{-bx}{\sqrt{\alpha}}}}{\sqrt{\pi}}\int_{b\sqrt{t}-x\eta}^{\infty} e^{-\gamma^2}\,d\gamma\right) \tag{j}$$

8.35. A careful examination of this expression is worth while to be sure that it is the desired solution. For $t = 0$ (*i.e.*, $\eta = \infty$) and $x \neq 0$ the lower limit of the first integral becomes ∞, hence the integral vanishes; in the second integral it becomes $-\infty$, giving a value of $\sqrt{\pi}$ to the integral. Hence, for $t = 0$ we have $T = 0$, as it should be for all cross sections of the rod except the heated end. Since both integrals have the same limiting value as $x \to 0$, this gives the right temperature for the end, *viz.*, $T = T_s$. Both integrals vanish for $t = \infty$, and thus, for the steady state, we have the result deduced in Secs. 3.5 and 3.7,

$$T = T_s e^{-bx/\sqrt{\alpha}} \tag{a}$$

From the value for b^2 given in (3.6f), *viz.*, $\alpha hp/kA$, we see that b^2 is very small if the emissivity is very small. Setting $b^2 = 0$ in (8.34j), we get

$$T = T_s\left(1 + \frac{1}{\sqrt{\pi}}\int_{x\eta}^{\infty} e^{-\gamma^2}\,d\gamma - \frac{1}{\sqrt{\pi}}\int_{-x\eta}^{\infty} e^{-\gamma^2}\,d\gamma\right) \tag{b}$$

which is readily seen to be identical with the results of Sec. 7.14 for the linear flow of heat in an infinite body.

8.36. Problems

1. A wrought-iron ($k = 0.144$, $\alpha = 0.173$ cgs) rod 1 cm in diameter and 1 m long is shielded with an impervious covering and subjected to temperatures 0°C and 100°C at its ends, until a steady state is reached. The covering is then removed and the rod placed in close contact at its ends with two long similar rods at zero, the temperature of the air being zero also. If h is 0.0003 cgs, what will be the temperature at the middle of the meter rod after 15 min (cf. Problem 6, Sec. 7.10)? *Ans.* 13.5°C

2. Show that Case IV can also be applied to this problem of the radiating rod.

CHAPTER 9

FLOW OF HEAT IN MORE THAN ONE DIMENSION

In this chapter we shall consider a few of the many heat-conduction problems involving more than one dimension. In particular we shall take up the case of the radial flow of heat, including heat sources, "cooling of the sphere," and cylindrical-flow problems; also, the general case of three-dimensional conduction.

CASE I. RADIAL FLOW. INITIAL TEMPERATURE GIVEN AS A FUNCTION OF THE DISTANCE FROM A FIXED POINT

9.1. This is the case analogous to the first discussed under linear flow in Chap. 7, but with the essential difference that the isothermal surfaces instead of being plane are here spherical. In the discussion of the steady state for radial flow (Sec. 4.5), we had occasion to express Fourier's equation in terms of the variable r, finding that

$$\nabla^2 T = \frac{1}{r}\frac{\partial^2(rT)}{\partial r^2} \tag{a}$$

the partial notation being used here to show differentiation with respect to r alone, T now depending on t as well; thus, the fundamental equation becomes

$$\frac{\partial T}{\partial t} = \frac{\alpha}{r}\frac{\partial^2(rT)}{\partial r^2} \tag{b}$$

or

$$\frac{\partial(rT)}{\partial t} = \alpha\frac{\partial^2(rT)}{\partial r^2} \tag{c}$$

The solution of our problem must satisfy this equation, and the boundary condition

$$T = f(r) \qquad \text{when } t = 0 \tag{d}$$

Let

$$u = rT \tag{e}$$

and our differential equation (c) reduces to

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial r^2} \tag{f}$$

where $\qquad u = rf(r) \qquad$ when $t = 0 \qquad (g)$

and $\qquad u = 0 \qquad$ at $r = 0 \qquad (h)$

u being always positive if T is taken as positive. But the solution of (f) under these conditions will be identical to that for the case of linear flow with one face at zero, treated in Sec. 7.12. Using, as in this case, λ as the variable of integration, and remembering that when $t = 0$

$$u = \lambda f(\lambda) \tag{i}$$

we have the temperature at any distance r from the point, given, from (7.12d), by the equation

$$u = rT = \frac{\eta}{\sqrt{\pi}}\left[\int_0^\infty \lambda f(\lambda)e^{-(\lambda-r)^2\eta^2}\,d\lambda - \int_0^\infty \lambda f(\lambda)e^{-(\lambda+r)^2\eta^2}\,d\lambda\right] \tag{j}$$

With the substitutions

$$\beta \equiv (\lambda - r)\eta \quad \text{or} \quad \lambda = \frac{\beta}{\eta} + r$$

and

$$\beta' \equiv (\lambda + r)\eta \quad \text{or} \quad \lambda = \frac{\beta'}{\eta} - r \tag{k}$$

this becomes

$$T = \frac{1}{r\sqrt{\pi}}\left[\int_{-r\eta}^{\infty}\left(\frac{\beta}{\eta} + r\right)f\left(\frac{\beta}{\eta} + r\right)e^{-\beta^2}\,d\beta - \int_{r\eta}^{\infty}\left(\frac{\beta'}{\eta} - r\right)f\left(\frac{\beta'}{\eta} - r\right)e^{-\beta'^2}\,d\beta'\right] \tag{l}$$

9.2. If the initial temperature is a constant, T_0, within a sphere of radius R in the infinite solid, and zero everywhere outside, the subsequent temperatures are given from (9.1j) by

$$T = \frac{T_0\eta}{r\sqrt{\pi}}\left[\int_0^R \lambda e^{-(\lambda-r)^2\eta^2}\,d\lambda - \int_0^R \lambda e^{-(\lambda+r)^2\eta^2}\,d\lambda\right] \tag{a}$$

or, from (9.1l), by

$$T = \frac{T_0}{r\sqrt{\pi}}\left[\int_{-r\eta}^{(R-r)\eta}\left(\frac{\beta}{\eta}+r\right)e^{-\beta^2}\,d\beta - \int_{r\eta}^{(R+r)\eta}\left(\frac{\beta}{\eta}-r\right)e^{-\beta^2}\,d\beta\right] \tag{b}$$

This gives T directly for all points save $r = 0$, where it becomes

indeterminate and must then be evaluated by differentiation. This gives for the center

$$T_c = T_0 \left[\Phi(R\eta) - \frac{2R\eta}{\sqrt{\pi}} e^{-R^2\eta^2} \right] \tag{c}$$

APPLICATIONS

9.3. The Cooling of a Laccolith. By means of equation (9.2*b*) we can solve a problem of interest to geologists, *viz.*, that

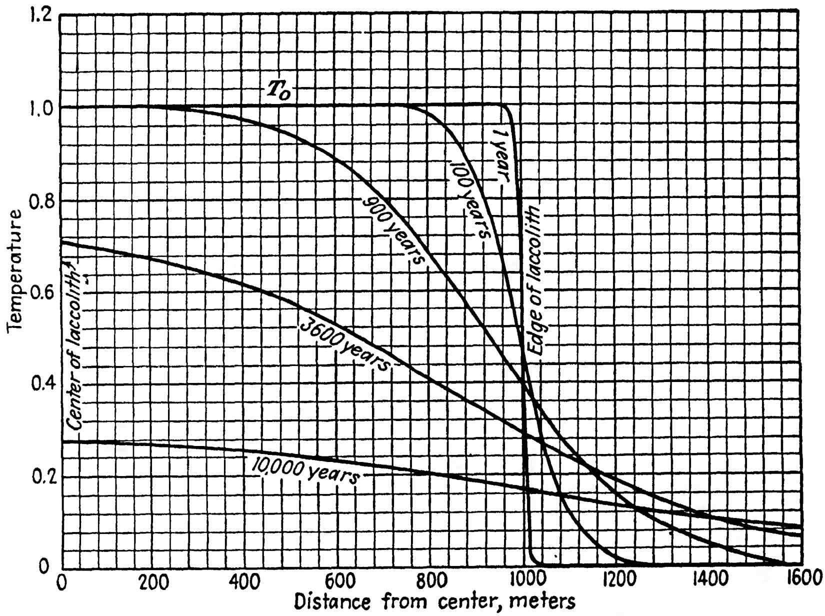

FIG. 9.1. Computed temperature curves for a laccolith 1,000 m in radius, which has been cooling from an initial temperature T_0 for various periods of time. A point 5 m from the boundary surface would reach its maximum temperature in about 100 years, while at 100 m the maximum would not be reached for over 1,000 years.

of the cooling of a laccolith. This is a huge mass of igneous rock, more or less spherical or lenticular in shape, which has been intruded in a molten condition into the midst of a sedimentary rock, *e.g.*, limestone. The importance of the formation, from a geological standpoint, lies in the fact that ores are frequently found in the region immediately adjoining the original surface of the laccolith, and the conditions and time of cooling of the

igneous mass would naturally have a bearing on any explanation of the deposit of such ores.

The temperature curves given in Fig. 9.1 were computed for the following conditions: radius R of laccolith, 1,000 m; diffusivity = 0.0118 cgs. (Kelvin's estimate. This is also not far from the mean of the values for granite and limestone; the medium must here be assumed to be uniform.) The initial temperature of the igneous rock is taken as T_0, probably between 1000 and 2000°C, while the surrounding rock is assumed at zero.

The conclusions to be drawn from the curves are (1) that the cooling is a very slow process, occupying tens of thousands of years; (2) that the boundary-surface temperature quickly falls to half* the initial value and then cools only slowly, and also that for a hundred or more years there is a large temperature gradient over only a few meters and a very slow progress of the heat wave; (3) that the maximum temperature in the limestone, or the crest (so to speak) of the heat wave, travels outward only a few centimeters a year. The mass behind it will then suffer a contraction as soon as it begins to cool, and the cracking and introduction of mineral-bearing material† is doubtless a consequence of this.

9.4. Problems

1. Molten copper at 1085°C is suddenly poured into a spherical cavity in a large mass of copper at 0°C. If the radius of the cavity is 20 cm, find the temperature at a point 10 cm from the center after 5 min. Also, solve for center. Neglect latent heat of fusion and assume $k = 0.92$, $\alpha = 1.133$ cgs.

Ans. 103°C; center, 109°C

2. Show that

$$T = \frac{T_s R}{r} \{1 - \Phi[(r - R)\eta]\} \ddagger \qquad \text{for } r \gtreqless R \qquad (a)$$

is a solution of the problem of the temperature in an infinite medium, initially at zero, which has a spherical cavity of radius R with surface kept at T_s from time $t = 0$.

SUGGESTION: Show that $u = rT$ is a solution of (9.1f) and satisfies the boundary conditions: $u = RT_s$ at $r = R$; $u = 0$ at $r = \infty$; $u = 0$ when $t = 0$.

* The temperature of the boundary surface for the first hundred years or so could best be estimated from (7.17d). The error introduced by assuming the diffusivities to be the same becomes less and less as the cooling proceeds.

† See Leith and Harder[84] and Jones.[73]

‡ We are indebted to Professor Felix Adler for pointing out certain features of this solution. See Carslaw and Jaeger.[37a, p. 209]

3. Show, by evaluating $\partial T/\partial r$ from (*a*), that the rate of heat inflow into the medium at $r = R$ in Problem 2 is

$$q = 4\pi kRT_s\left(1 + \frac{R}{\sqrt{\pi\alpha t}}\right) \tag{b}$$

4. In the application of Sec. 4.10 find approximately how long it will take for the steady state to be established. In doing this, calculate the rate of heat inflow after 1 week, 1 month, 3 months, 1 year, and 10 years, assuming a constant surface temperature of 200°F below the initial lava ($k = 1.2$, $\alpha = 0.03$ fph) temperature.
Ans. 24,200, 17,870, 15,450, 13,750, 12,600 Btu/hr. Steady-state rate is 12,050 Btu/hr

CASE II. HEAT SOURCES AND SINKS

9.5. Point Source. If Q units of heat are suddenly developed at a point in the interior of a solid that is everywhere else at zero, a radial flow will at once take place and the temperature at any point for any subsequent time can be found in terms of the time and the distance from this center. This case is analogous to that discussed in Sec. 8.3, where we had a linear flow from an instantaneous heat source located in a plane of infinitesimal thickness. Just as in this case, too, we can deduce the solution by a special application of a more general one. For if in (9.2*a*) we let the radius R of the spherical region, which is initially at constant temperature T_0, become vanishingly small, while its initial temperature is correspondingly increased so as to make the amount of heat finite, we shall have a solution of the present problem.

To get this, put

$$Q = T_0 c\rho \tfrac{4}{3}\pi R^3 \tag{a}$$

as the amount of heat in a very small sphere of radius R, and substitute the value of T_0 deduced from this in (9.2*a*). Then,

$$T = \frac{3Q\eta}{4c\rho R^3 \pi^{1/2} r}\left(\int_0^R \lambda e^{-(\lambda - r)^2\eta^2}\,d\lambda - \int_0^R \lambda e^{-(\lambda + r)^2\eta^2}\,d\lambda\right) \tag{b}$$

Now we may write

$$e^{-(\lambda - r)^2\eta^2} = e^{-\lambda^2\eta^2} e^{2\lambda r\eta^2} e^{-r^2\eta^2} \tag{c}$$

$$= \left(1 - \lambda^2\eta^2 + \frac{\lambda^4\eta^4}{2!} - \cdots\right)\left(1 + 2\lambda r\eta^2 + \frac{4\lambda^2 r^2\eta^4}{2!} + \cdots\right)e^{-r^2\eta^2} \tag{d}$$

since

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots \tag{e}$$

We can see by inspection the similar expression for $e^{-(\lambda+r)^2\eta^2}$. Since λ is a very small quantity in this integration, being confined to the limits 0 and R, (d) simplifies to

$$(1 + 2\lambda r\eta^2)e^{-r^2\eta^2} \tag{f}$$

the effect of the other terms vanishing in the limit as $R \to 0$, as may be readily seen on inspection of (g) following:

Then, (b) becomes

$$T = \frac{3Q\eta}{4c\rho R^3\pi^{3/2}r} e^{-r^2\eta^2}\left[\int_0^R \lambda(1 + 2\lambda r\eta^2)\,d\lambda - \int_0^R \lambda(1 - 2\lambda r\eta^2)\,d\lambda\right] \tag{g}$$

$$= \frac{3Q\eta}{4c\rho R^3\pi^{3/2}r} e^{-r^2\eta^2} \frac{4R^3 r\eta^2}{3} \tag{h}$$

$$= Q\frac{\alpha}{k}\left(\frac{\eta}{\sqrt{\pi}}\right)^3 e^{-r^2\eta^2} = S\left(\frac{\eta}{\sqrt{\pi}}\right)^3 e^{-r^2\eta^2} \tag{i}$$

By the same reasoning used in deriving (8.3g) we can write with the aid of (h) and (i) the expression for the temperature at a distance r from a permanent source releasing Q' units of heat per second (or hour if in fph), starting t sec (hr) ago, as

$$T = \frac{Q'}{8c\rho\pi^{3/2}\alpha^{3/2}} \int_0^t e^{\frac{-r^2}{4\alpha(t-\tau)}} (t - \tau)^{-3/2}\,d\tau \tag{j}$$

which reduces, on putting $\beta \equiv r/2\sqrt{\alpha(t - \tau)}$, to

$$T = \frac{Q'}{2c\rho\pi^{3/2}\,\alpha r} \int_{r\eta}^{\infty} e^{-\beta^2}\,d\beta = \frac{Q'}{4\pi k r}\,\frac{2}{\sqrt{\pi}} \int_{r\eta}^{\infty} e^{-\beta^2}\,d\beta \tag{k}$$

or, writing $S' = Q'/c\rho$,

$$T = \frac{S'}{2\pi^{3/2}\alpha r} \int_{r\eta}^{\infty} e^{-\beta^2}\,d\beta \tag{l}$$

If we put $t = \infty$ in the last equations, we have

$$T = \frac{S'}{4\pi\alpha r} = \frac{Q'}{4\pi c\rho\alpha r} = \frac{Q'}{4\pi k r} \tag{m}$$

as the temperature for the steady state in an infinite solid where Q' units of heat are released per unit of time, at a point [cf. (4.5p), noting that here q and Q' have the same value].

If the permanent source, instead of being of constant strength $Q'/c\rho$, is of variable strength $f(t)$, (j) becomes

$$T = \frac{1}{8\pi^{3/2}\alpha^{3/2}} \int_0^t f(\tau) e^{\frac{-r^2}{4\alpha(t-\tau)}} (t - \tau)^{-3/2}\, d\tau \tag{n}$$

or

$$T = \frac{1}{2\pi^{3/2}\alpha r} \int_{r\eta}^{\infty} f\left(t - \frac{r^2}{4\alpha\beta^2}\right) e^{-\beta^2}\, d\beta \tag{o}$$

Equation (9.5i) shows that T has a value different from zero in all parts of space even when t is exceedingly small, or, in other words, that heat is propagated apparently with an infinite velocity. As a matter of fact, the heat disturbance is undoubtedly transmitted with great rapidity through the medium, although it is continually losing so much energy to this medium, which it has to heat up as it passes through, that the actual amount of heat traveling any appreciable distance from the source in a very short time is very small.

9.6. With (9.5i) derived, it may be instructive to reverse the process and show that it is our desired solution. To do this we must show that it satisfies (9.1c) and the boundary conditions

$$T = 0 \quad \text{when } t = \infty \tag{a}$$
$$T = 0 \quad \text{when } t = 0 \text{ save at } r = 0 \tag{b}$$

and also the condition that the total amount of heat at any time shall equal Q.

Differentiation gives

$$\frac{\partial(rT)}{\partial t} = \left(-\frac{3}{2t} + \frac{r^2}{4\alpha t^2}\right) rT \tag{c}$$

$$\frac{\partial(rT)}{\partial r} = \left(\frac{1}{r} - \frac{r}{2\alpha t}\right) rT \tag{d}$$

$$\frac{\partial^2(rT)}{\partial r^2} = \left(-\frac{3}{2\alpha t} + \frac{r^2}{4\alpha^2 t^2}\right) rT \tag{e}$$

showing that (9.1c) is satisfied. That conditions (a) and (b) are fulfilled may be shown if we rewrite that part of (9.5i) con-

taining t,

$$\frac{1}{t^{3/2}e^{b/t}} = \frac{1}{(t^{3/2})[1 + (b/t) + (b^2/2!t^2) + \cdots]} \tag{f}$$

The denominator is seen to be infinite for $t = 0$ or ∞; hence, (9.5i) vanishes for each of these values. As to the last condition, the total amount of heat is given by

$$\int_0^\infty \rho c T 4\pi r^2\, dr = \int_0^\infty Q \left(\frac{\eta}{\sqrt{\pi}}\right)^3 e^{-r^2\eta^2} \cdot 4\pi r^2\, dr \tag{g}$$

If we put

$$\gamma \equiv r\eta \tag{h}$$

the second member becomes

$$\frac{4Q}{\sqrt{\pi}} \int_0^\infty e^{-\gamma^2}\gamma^2\, d\gamma \tag{i}$$

which (Appendix C) is equal to Q.

9.7. The time t_1 at which T reaches its maximum value is given by differentiating (9.5i) and equating to zero. This gives

$$t_1 = \frac{r^2}{6\alpha} \tag{a}$$

The corresponding temperature is

$$T_1 = \left(\frac{1}{\sqrt{2/3\pi e}}\right)^3 \frac{Q}{c\rho r^3} \tag{b}$$

9.8. Line Source. Point Source in a Plane Sheet. A line source may be thought of as a continuous series of point sources along an infinite straight line. The magnitude of each such point source would be $Q\, dz$, where Q is the heat released per unit length of line. Similarly, the strength is $S\, dz$. To get the effect of such an instantaneous line source in an infinite medium, initially at zero, at a point distant r from the line, we sum the effects of terms like (9.5i) and get

$$T = \int_{-\infty}^{\infty} S \left(\frac{\eta}{\sqrt{\pi}}\right)^3 e^{-(r^2+z^2)\eta^2}\, dz$$
$$= S \left(\frac{\eta}{\sqrt{\pi}}\right)^3 e^{-r^2\eta^2} \int_{-\infty}^{\infty} e^{-z^2\eta^2}\, dz = \frac{S\eta^2}{\pi} e^{-r^2\eta^2}\text{*} \tag{a}$$

* It will appear in Sec. 9.41 that (a) and also (8.3e) and (9.5i) are special cases of (9.41e). It may also be pointed out that (8.3e) is readily obtainable from (a) as the summation of the effects of a continuous distribution of line sources in a plane.

The flow of heat from a point source in a thin plane sheet or lamina, if there is no radiation or other loss from the sides, may be considered as a special case of line source, perpendicular to the plane, since the heat flow is all normal to such line source, *i.e.*, radially in the plane. Equation (a) applies if we divide the actual amount of heat released at the point by the *thickness* of the sheet, so as to get Q (or S) for unit thickness, *i.e.*, per unit length of line source.

If the line source, or the point source in a plane, is a permanent one starting at zero time, and if the plane or medium is everywhere initially at zero temperature, the temperature at any later time t at any point may be written at once as

$$T = \frac{S'}{4\pi\alpha}\int_0^t e^{-r^2/4\alpha(t-\tau)}(t-\tau)^{-1}\,d\tau \tag{b}$$

or, putting

$$\beta \equiv \frac{r}{2\sqrt{\alpha(t-\tau)}} \tag{c}$$

we have

$$T = \frac{S'}{2\pi\alpha}\int_{r\eta}^{\infty}\frac{e^{-\beta^2}}{\beta}\,d\beta \equiv \frac{S'}{2\pi\alpha}I(r\eta) = \frac{Q'}{2\pi k}I(r\eta) \tag{d}$$

where Q' is the number of heat units released per unit of time per unit length of the line source. For values of this integral see Appendix F.

It is of interest to calculate the rate of heat outflow for any radius r_1. To do this we must first differentiate (d), using Appendix K, and get

$$\frac{\partial T}{\partial r}\left(=\frac{\partial T}{\partial(r\eta)}\frac{\partial(r\eta)}{\partial r}\right) = \frac{-Q'}{2\pi k}\frac{e^{-r_1^2\eta^2}}{r_1\eta}\eta \tag{e}$$

Then, the rate of heat outflow per unit length of cylinder at any radius r_1 would be

$$q = 2\pi r_1 k\,\frac{Q'e^{-r_1^2\eta^2}}{2\pi k r_1} = Q'e^{-r_1^2\eta^2} = S'c\rho e^{-r_1^2\eta^2} \tag{f}$$

9.9. Synopsis of Source and Sink Equations. From Secs. 8.3, 9.5, and 9.8 we may write the general heat-source equation

$$T = \frac{S\eta^n}{\pi^{n/2}}e^{-r^2\eta^2}* \tag{a}$$

* See Jahnke and Emde[66, pp. 47–51 addenda] for graphs of this function.

where T is the temperature in a medium initially at zero at distance r from an instantaneous source of strength S at time t after its release. $n = 1$ for the linear-flow case (Sec. 8.3), 2 for the two-dimensional case (Sec. 9.8), and 3 for the three-dimensional case (Sec. 9.5). The three equations (a) are sometimes referred to as the fundamental solutions of the heat conduction equation.

For a permanent source the temperature at time t after its start is given by

$$T = \frac{S'r^{(2-n)}}{2\pi^{n/2}\alpha} \int_{r\eta}^{\infty} \beta^{(n-3)}e^{-\beta^2}\,d\beta = \frac{Q'r^{(2-n)}}{2\pi^{n/2}k} \int_{r\eta}^{\infty} \beta^{(n-3)}e^{-\beta^2}\,d\beta \qquad (b)$$

For the evaluation of this integral see Appendixes B, D, and F.* Many illustrations of its use will be found in the following applications, particularly in Secs. 9.11–9.12. Q' is expressed in Btu/hr or cal/sec for the three-dimensional case; in Btu/hr per ft length or cal/sec per cm length for the line source or sink; and in Btu/(hr)(ft²) or cal/(sec)(cm²) for the plane source or sink.

An inspection of the three integrals involved in (b) will show that the only case in which there is a steady state is for $n = 3$. For the other two cases, as t approaches infinity, T increases indefinitely. For points very close to the plane source the temperature is roughly proportional to the square root of the time, as shown in (9.12e), while for the line source the rise is slower. Further study of (b) will show that the plane source is the only case of the three that gives a finite temperature for $r = 0$.

If there are a number of sources in an infinite medium, the temperature at any point is the sum of the effects due to each source separately, making use of a principle we have already applied many times.

An inspection of the way in which (9.5n) and (9.5o) are obtained from (9.5j) and (9.5l) will show at once how to modify (b) to fit the case where a permanent source, instead of having a constant strength S', is of variable strength $f(t)$.

For an instantaneous source the time t_1 at which the maximum temperature is reached at a point r distant, is, as deter-

* See also (9.12d) for the integration for the plane source.

mined by methods similar to those of Sec. 9.7,

$$t_1 = \frac{r^2}{2n\alpha} \tag{c}$$

while the corresponding value of this maximum temperature is

$$T_1 = \frac{S}{(r\sqrt{2\pi e/n})^n} \tag{d}$$

where n in all cases has the values given above.

APPLICATIONS

9.10. Subterranean Sources and Sinks; Geysers. The foregoing source and sink equations have many interesting applications, of which we shall consider a few in this and the following sections.

1. Suppose heat is applied electrically or otherwise at the bottom of a drill hole or well—perhaps in an attempt to increase the flow of oil—at the rate of 360,000 Btu/hr. Take the thermal constants of the rock as $k = 1.2$, $c = 0.22$, $\rho = 168$, $\alpha = 0.032$ fph. What temperature rise might be expected at a distance of 15 ft from the source after 1,000 hr of heating? Using (9.5k) or (9.9b), we have

$$T = \frac{360{,}000}{2\pi^{3/2} \times 1.2 \times 15} \int_{15/2\sqrt{32}}^{\infty} e^{-\beta^2}\,d\beta$$
$$= 1{,}592[1 - \Phi(1.33)] = 96°\text{F} \tag{a}$$

2. It was indicated in Sec. 4.10 how calculations could be made on geysers, assuming that all the heat was supplied at the bottom of the tube. It is probable, however, that cylindrical flow more nearly fits the average case, and we shall make use in this connection of (9.8d) or (9.9b). Assume that in an old lava bed (use $k = 4.8 \times 10^{-3}$, $c = 0.22$, $\rho = 2.7$, $\alpha = 8.1 \times 10^{-3}$ cgs) at 400°C we have a geyser tube equivalent to a circular hole of 30 cm radius and of such depth that the average water temperature at eruption is 140°C. Equation (9.8d) gives the relation between the temperature T, in a medium at zero, at a distance r from a permanent line source or sink of strength S'

(per unit length) and the time t since it started. In handling the problem we shall shift the temperature scale by 400°C and overlook the minus signs this involves.

We need not inquire for the moment what happens inside $r = 30$ cm but will merely ask what constant strength of source S' will result in a temperature T of 260°C (*i.e.*, 400 − 140) at $r = 30$ cm, after a specified time that we shall take in this case as 100 years, or 3.156×10^9 sec. Then, $r/2\sqrt{\alpha t} = 2.96 \times 10^{-3}$; thus, we have

$$260 = \frac{S'}{2\pi \times 0.0081} \int_{2.96 \times 10^{-3}}^{\infty} \frac{e^{-\beta^2}}{\beta}\, d\beta \tag{b}$$

From Appendix F the integral evaluates as 5.54; thus, $S' = 2.39$. This gives $Q'(= S'c\rho) = 1.42$ cal/sec per cm length of tube. If the water enters the geyser tube at 20°C, the heat required per cm length of tube to start an eruption would be approximately $\pi \times 30^2 \times 120 = 3.39 \times 10^5$ cal, giving a period of 2.4×10^5 sec or 67 hr. For 10,000 years this would work out to 94 hr.*

We must now examine a little more closely just what we have done in this solution. Equation (9.8*e*) gives the temperature gradient at a distance r_1 from the line source at time t, and (9.8*f*) the rate of heat outflow or inflow through the cylindrical surface of radius r_1. It is evident then that the problem of the line source emitting or absorbing Q' heat units per unit time per unit length of source is, *for values of r equal to or greater than r_1*, equivalent to that of a cylindrical source of radius r_1 emitting $Q'e^{-r_1^2\eta^2}$ heat units per unit time per unit length of cylinder. In other words, we may regard (9.8*e*) and (9.8*f*) as a boundary condition† for the medium ($r \geqq r_1$) that is the

* It is to be noted that these two calculations of period really apply to two different geysers. The equations apply only to a permanent source or sink of constant strength, and so what has been calculated here is not the increase in period of a single geyser but the period of another of such constant strength of sink (somewhat smaller than the other) that after 10,000 years the temperature at $r = 30$ cm is 260°C below the initial value. The increase in period of a single geyser would certainly be of this order of magnitude, but the exact calculations would be difficult.

† Somewhat this same reasoning has already been used in the footnote of Sec. 7.21.

same for either the line or cylindrical source. (The other boundary conditions are $T = 0$ everywhere in the medium at $t = 0$, and $T = 0$ at infinity.)

We see then that we have really solved the problem for an ideal geyser whose rate of heat inflow from the surrounding medium is determined by (9.8*f*). However, if we calculate (9.8*f*) for 1 year we have, since here $\eta^2 = 1/(1.02 \times 10^6)$,

$$q = Q'e^{-900\eta^2} = Q'(1 - 9 \times 10^{-4}) \qquad (c)$$

This means that for $r = 30$ cm and for values of t greater than 1 year the rate of heat inflow would differ from Q' by less than 0.1 per cent.

3. As a third example of the use of source and sink equations we shall inquire in connection with the application of Sec. 4.10 approximately how long before the condition indicated there, *i.e.*, the steady state, might be reached. Accordingly, we shall calculate with the aid of (9.5*k*) or (9.9*b*) what temperatures would be found 4 ft away from a permanent source (or sink) generating (or absorbing) 12,050 Btu/hr after 1,000 hr. Using $k = 1.2$, $\alpha = 0.03$ fph, we have

$$T = \frac{12{,}050}{2 \times 1.2 \times \pi^{3/2} \times 4} \int_{4/2\sqrt{30}}^{\infty} e^{-\beta^2}\,d\beta = 200[1 - \Phi(0.365)] = 121°\text{F} \qquad (d)$$

This means that the temperature at 4 ft distance is 121°F cooler than the original rock temperature of 500°F. In 100,000 hr the value is 192°F or within 8°F of the final temperature. We may then conclude that anything approaching the steady state in this case would take ten years or more. It is to be noted that, until the steady state is reached, the same type of (justifiable) approximation is involved here as was investigated in the preceding paragraph.*

9.11. Heat Sources for the Heat Pump. The heat pump is one of the newest and most interesting developments in air conditioning; it serves the dual purpose of heating a building

* See Sec. 9.4, Problem 4, for a treatment of this problem under slightly different assumptions.

in winter and cooling it in summer. Working in the reversed thermodynamic cycle, like the ordinary electric refrigerator, it absorbs heat from a cold body or region, adds to it by virtue of the energy that must be supplied to operate the machinery, and supplies this augmented energy to the building that is being heated (winter operation). This energy may be three or four times the electrical energy required and its operation is accordingly cheaper, in this ratio, than plain electric heating.

In the operation of the heat pump for heating in winter it is necessary to have some outside medium from which heat can be absorbed. In some installations the outside air is used, in others well water or running water; but in an increasing number of cases arrangement is made to abstract the heat from the ground* itself. This means the installation of a considerable length of pipe, small or large, in good thermal contact with the ground below frost line or with the underlying rock, in which fluid can be circulated. It is highly desirable to be able to calculate the temperatures that might be expected in such circulating fluid as dependent on the rate of heat withdrawal, the time since the start of the operation, and the thermal constants of the soil or rock, which is initially at a known temperature—assumed uniform but actually varying slightly, of course, with depth.

This is essentially the problem of the line sink, and we shall solve two special cases. The first is to calculate the temperatures that might be expected in an 8-in.† diameter pipe if 50 Btu/hr per linear ft of pipe is abstracted from it. We shall use as constants for the soil or rock $k = 1.5$ (high!), $c = 0.45$, $\rho = 103$, $\alpha = 0.0324$ fph. Temperatures are to be calculated after 1 week, 1 month, and 6 months of operation at this average rate of heat withdrawal.

Using (9.8*d*), we have for 1 week or 168 hr

$$T = \frac{50}{2\pi \times 1.5} \int_{\frac{0.333}{2\sqrt{0.0324 \times 168}}}^{\infty} \frac{e^{-\beta^2}}{\beta}\, d\beta = 5.31 I(0.0715) \qquad (a)$$

* See E. N. Kemler.[74]

† The pipe dimensions given in this and the following sections are outside diameters. For simplicity, round numbers, rather than standard pipe sizes, are used in the illustrations.

This gives, with the aid of Appendix F, $T = 12.5°F$ below the initial soil temperature of perhaps 50°F. The values for 1 month and 6 months are 16.4 and 21.4°F, respectively. For a 2-in. pipe four times as long (*i.e.*, same surface) with the same total heat withdrawal we have, for 1 week,

$$T = \frac{12.5}{2\pi \times 1.5} \int_{\frac{0.0833}{2\sqrt{0.0324 \times 168}}}^{\infty} \frac{e^{-\beta^2}}{\beta} d\beta = 1.33I(0.0179) \qquad (b)$$

This gives a value of 5.02°F below initial ground temperatures, with values of 5.96 and 7.15°F for 1 month and 6 months. Since it is desirable to have a heat source that is no colder than necessary, it is evident that, for a given exposed surface, the long small pipe is better than the shorter large one.

In applying the line source equation (9.8*d*) to this problem we are making certain assumptions:

1. The pipe must be long enough so that the heat flow is all normal to its length, *i.e.*, radial. This would probably be approximately true in most cases.

2. Since we really have a cylindrical source of radius r_1 instead of a true line source, we must, according to the considerations brought out in the latter part of Sec. 9.10, No. 2, assume that the heat is absorbed, not at the rate Q', but at $Q'e^{-r_1^2\eta^2}$. For the 2-in. pipe above treated this means that the absorption rate should start at zero, rise to $0.8Q'$ in a quarter of an hour, $0.95Q'$ in one hour, and $0.99Q'$ in five hours. The difference between the effect of this and a uniform rate Q' from the start is inconsequential after the first half day's run with a small pipe, but this period would be considerably longer for a large one.

Subject to the above conditions, (9.8*d*) would give, for $r \geqq r_1$, temperatures due to a single pipe in an infinite medium initially everywhere at zero. If the medium is, say, 30° above zero initially, this amount should be added to all temperatures calculated with this equation; *i.e.*, shift the scale as indicated in the above examples. If the initial temperature varies with the distance from the pipe, the effect of the pipe should be added to the changes which would take place with time due to the initial gradients, *i.e.*, we use the sum of two separate solutions. If

there is more than one pipe the temperature at any point, *e.g.*, the surface of a pipe, would be the sum of the effects at that point of each pipe.

If the pipe or pipes are near a ground surface kept at zero, the problem may be solved by assuming, in addition, a (negative) image of the pipe(s) above the ground surface. [This is essentially the principle used in deriving (7.12*c*).] If instead the surface is impervious to heat, the solution would involve the assumption of a positive image (see Sec. 7.28, Prob. 6). If, as is usually the case, the surface undergoes seasonal temperature variations, the temperature at any point would be the sum of the effects due to the pipes with ground surface held at zero, plus the effect of the seasonal variation at the point.

If Q' is not constant but varies from month to month, the integral (9.8*d*) may be split into parts. If the effect is desired at the end of 3 months of operation, we use the sum of three integrals, with Q' in each case taken as the average for the corresponding month. The limits in each case would be determined by the *times since the particular interval under consideration began and ended.* A study of (8.13*d*) will aid in clarifying this point. Cases where the temperature varies markedly along the pipe would present some special difficulties. It is possible that the rigorous calculations of Kingston[77] on the cooling of concrete dams (Sec. 9.14) could be applied to this problem.

Some of these same considerations may be applied to the heat dissipation from underground power cables. However, the relatively shallow depth, as well as other conditions, may bring about an approximately steady state after a comparatively short time of operation.

9.12. Spherical and Plane Sources for the Heat Pump. While the long small buried pipe seems the most feasible ground source for the heat pump, a number of other forms have been suggested. One of these is the "buried cistern" or large roughly spherical cavity deep in the ground. We shall make some calculations for such a cavity of radius 5 ft, in soil of the same high-thermal-conductivity constants ($k = 1.5$, $\alpha = 0.0324$ fph) as used above. If we take the same rate of heat absorption as already used, *viz.*, 23.9 Btu/hr per ft² of surface (correspond-

ing to 50 Btu/hr per ft length of 8-in. pipe), we get

$$Q' = 23.9 \times 4\pi \times 25 = 7510 \text{ Btu/hr}$$

for the cavity. This corresponds to 150 ft of 8-in. pipe or 600 ft of 2-in. pipe. Using (9.9*b*) with $n = 3$ and $t = \infty$ (*i.e.*, $\eta = 0$), we have

$$T = \frac{Q'}{4\pi kr}\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-\beta^2}\,d\beta = \frac{Q'}{4\pi kr} \qquad (a)$$

for the steady state. This is the same as (4.5*p*), since under these conditions q and Q' have the same value. This gives, for $r = 5$, *i.e.*, the surface of the cavity, $T_s = 79.8°\text{F}$ below the initial temperature.

We shall now investigate conditions before the cavity reaches a steady temperature state. The exact solution of the problem* of what temperature on the surface of the cavity, as a function of time, will give a *uniform* rate of heat absorption of 7510 Btu/hr is not easy. We can, however, readily solve two problems closely related to this.

The first problem involves a uniform temperature of the surface of the cavity. Its solution is reached by a simple application of (9.4*b*). Using this and taking T_s as 79.8°F below the initial soil temperature, as used above for the steady state, we have the following values for q, the rate of heat inflow: 16,600 Btu/hr at the end of 1 week; 11,870 Btu/hr at the end of 1 month; and 9300 Btu/hr at the end of 6 months.

The second solution is somewhat more complicated. Here we shall use (9.9*b*) with $n = 3$, and differentiate it with respect to r to get the temperature gradient and corresponding rate of heat inflow for any radius r and time t. We must assume a particular value of Q', which we shall choose the same as that used above, *viz.*, 7510 Btu/hr. The corresponding value of T for the radius r in which we are particularly interested, *i.e.*, 5 ft, is obtained at once from (9.9*b*). The result of this calculation† will be a series of values of T_5 and q_5 for the cavity surface temperature and rate of heat inflow, for various times. If,

* In this connection see Carslaw.[27, p. 151]

† In this connection examine again the reasoning in Example 2 of Sec. 9.10.

then, the rate of heat inflow is made to vary with time as indicated by these values, the surface temperature will take the corresponding values. It is to be noted that this is a special series of values of T_5 and q_5 that is afforded by our point-heat-source theory. While neither this series nor the one given above may fit the actual case—of course, it must be remembered that the values can be adjusted to any scale by the proper choice of Q'—the two solutions together should enable one to furnish an approximate theoretical background for any practical case.

In reaching the second solution we first write (9.9*b*) for $n = 3$, which gives

$$T = \frac{Q'}{4\pi kr} \frac{2}{\sqrt{\pi}} \int_{r\eta}^{\infty} e^{-\beta^2}\, d\beta \tag{b}$$

We then differentiate it [see Appendix K, also (9.8*e*)] and get

$$\frac{\partial T}{\partial r} = \frac{Q'}{4\pi kr}\left(- \frac{2\eta}{\sqrt{\pi}} e^{-r^2\eta^2} - \frac{1}{r} \frac{2}{\sqrt{\pi}} \int_{r\eta}^{\infty} e^{-\beta^2}\, d\beta \right) \tag{c}$$

For 1 week or 168 hr, $\eta = 0.215$, giving $\partial T/\partial r = -8.2°\text{F/ft}$. This gives a rate of heat absorption at $r = 5$ ft of

$$q_5' = 4\pi \times 25 \times 1.5 \times 8.2 = 3870 \text{ Btu/hr}$$

The corresponding temperature is, from (*b*), $T = 10.3°\text{F}$ below the initial one of the surroundings. For 1 month these values are 6840 Btu/hr and 37.4°F, while for 6 months they are 7460 Btu/hr and 61.3°F below the initial value.

Another type of heat absorber that has been suggested is the plane. In its most feasible form this would probably be an array of pipes looped back and forth in a plane, the spacing being much less than would allow them to be considered independently as treated above. Putting $n = 1$ in (9.9*b*) we have, using Appendix B,

$$T = \frac{Q'r}{2k\sqrt{\pi}} \int_{r\eta}^{\infty} \frac{e^{-\beta^2}}{\beta^2}\, d\beta = \frac{Q'r}{2k}\left(\frac{e^{-r^2\eta^2}}{r\eta\sqrt{\pi}} - \frac{2}{\sqrt{\pi}} \int_{r\eta}^{\infty} e^{-\beta^2}\, d\beta \right) \tag{d}$$

For $r = 0$ this becomes

$$T = \frac{Q'}{2k\eta\sqrt{\pi}} = \frac{Q'\sqrt{\alpha t}}{k\sqrt{\pi}} \tag{e}$$

With $Q' = 23.9$ Btu/hr per ft² of surface, as used above, this gives $T = 21.0$°F below the initial temperature at the end of 1 week, 43.8°F after 1 month, and 107.3°F after 6 months. If such a plane absorber is located near the surface of the ground or below a basement floor, as has been suggested at times, the heat flow might become mostly a one-sided matter and, accordingly, the above temperatures would have to be almost doubled.

The relatively rapid lowering of temperature with time in these two latter heat absorbers (not considering the steady state that is eventually reached for the spherical cavity) is one of the factors that point to the long small ground pipe—perhaps in the form of one or more vertical "wells"—as perhaps the best type of absorber or heat source that has been suggested.*

9.13. Electric Welding. A welding machine joining the straight edges of two flat steel ($k = 0.11$, $c = 0.12$, $\rho = 7.8$, $\alpha = 0.118$ cgs) plates 8 mm (0.315 in.) thick uses 2000 cal per cm length of weld. What maximum temperature will be reached in the plate 5 cm (1.97 in.) from the weld and when?

Assuming that all the heat is retained in the plate, that half flows in each direction, and that it is generated effectively instantaneously, we have

$$Q \text{ (per cm}^2\text{ of the weld)} = \frac{2{,}000}{0.8} = 2{,}500$$

or $S = 2{,}670$. Then (9.9*d*), with $n = 1$, gives

$$T_1 = \frac{2{,}670}{5\sqrt{2\pi e}} = 129°\text{C} \qquad (a)$$

and, from (9.9*c*),

$$t_1 = \frac{25}{2 \times 0.118} = 106 \text{ sec} \qquad (b)$$

As a second example, consider a spot-welding operation where 2,400 watts for 2 sec generates 4,800 joules or 1146 cal at

* Consideration, however, should be given to the fact that, if more heat is taken from a system of deep vertical pipes in winter than is returned in summer, a progressive lowering of deep earth temperatures may result in the course of years—a situation that might not be remedied by conduction in from the surface in summer. This effect could be readily calculated for a period of years by using for Q' the average for the year. Because of the slowness with which the integral $I(x)$ increases this progressive lowering would not be a serious matter for a single pipe. It would, in any case, be markedly altered by even a small underground water movement.

a point in a steel plate 1.5 mm (0.059 in.) thick. What maximum temperature is reached 4 cm (1.57 in.) away from the point and when?

Using the above constants for steel, we find

$$Q = \frac{1{,}146}{0.15} = 7{,}630$$

(on the basis of unit thickness) and $S = 8{,}170$. Then using $n = 2$ in (9.9*c*) and (9.9*d*), we have

$$T_1 = \frac{8{,}170}{4^2 \pi e} = 59.9°\text{C}, \qquad t_1 = \frac{4^2}{4 \times 0.118} = 33.9 \text{ sec} \qquad (c)$$

It is evident that if these calculations are carried out for points very close to the weld, the temperatures arrived at would be far above the melting point of the metal. This simply means that this is not really an instantaneous source of heat, nor is the heat all delivered strictly at a line or point. Consequently, calculations cannot be made for such points with the equations used above.

From a conduction standpoint the generation of heat in electrical contacts may be considered as a special case of spot welding. For an approximate treatment we may assume that such a contact is frequently, if not generally, shaped like the frustum of a cone, with the heat generation at the tip. The cone can be considered as part of a sphere, the fraction being determined by the ratio of its solid angle to 4π. Temperatures resulting from the sudden generation of a small amount of heat at the tip can then be calculated from (9.5*i*) or (9.9*a*), or, for maximum values (9.9*d*). It is evident, however, that in using these equations the amount of heat Q must be taken as the heat generated at the contact multiplied by the ratio of 4π to the solid angle of the cone. See footnote to Sec. 4.12, Problem 6.

9.14. Cooling of Concrete Dams. Because of the heat released in the hydration of cement large masses of concrete, as in dams, will rise many degrees in temperature unless special cooling is provided. Without such artificial cooling the temperature rise might be 50°F or more; the heat would require years to dissipate and the final inevitable contraction would

cause extensive cracking. Rawhouser[117] has described the methods used in cooling Boulder, Grand Coulee, and other dams, and their results. This is accomplished by embedding 1 in. (o.d.) pipes in the concrete about 5 or 6 ft apart and circulating cold water through them for a month or two, beginning as soon as the concrete is poured.

The problems involved in such conduction cooling have been extensively studied by the U.S. Bureau of Reclamation engineers.[22]* The three following calculations are, by comparison, crude and simple but not without interest since they arrive at results of the right order of magnitude by relatively simple means. We shall assume the pipes 6 ft apart and staggered so that each pipe cools a cylinder of hexagonal section of area 31.2 ft^2, equivalent to a circle of radius 3.15 ft. Take as thermal constants of the concrete $k = 1.4$, $c = 0.22$, $\rho = 154$, $\alpha = 0.041$ fph, and assume that the heat released by hydration is 6 cal/gm or 10.8 Btu/lb, which would cause an adiabatic temperature rise of 49°F. This hydration heat amounts to 1663 Btu/ft^3; thus, each foot of pipe must carry away 51,900 Btu.

Our first calculation will be only a rough approximation. Assume that the heat is released at a uniform rate and carried away as released (*i.e.*, steady state) and that the mass of concrete averages 15°F in temperature above the cooling pipe. Furthermore, since for such a small pipe (radius 0.0417 ft) the temperature gradient is much the largest near the pipe, we shall arbitrarily assume that the concrete temperature remains uniformly 15°F above the pipe at distances greater than 1 ft from the pipe. We then get with the aid of (4.6*f*), as the heat loss per foot of pipe,

$$\frac{2\pi \times 1.4 \times 15}{(2.303 \log_{10} 1/0.0417)} = 41.5 \text{ Btu/hr}$$

This would involve a total time of the order, of 1,250 hr or 52 days for the dissipation of all the heat.

Perhaps a better approximation is afforded by the following treatment: Suppose the hydration heat of 1663 Btu/ft^3 is released at a uniform rate so that the process is completed in

* See also Glover,[47] Rawhouser,[117] Kingston,[77] and Savage.[121]

1,250 hr, which means a rate of heat development $q_v = 1.33$ Btu/(ft^3)(hr). Then for a foot length of pipe the rate of heat flow through any annulus will be determined by the steady-state equation

$$q = q_v(\pi R^2 - \pi r^2) = \frac{k2\pi r \Delta T}{\Delta r} \qquad (a)$$

This means that for a cylinder of external radius R the heat that flows through any annulus of average radius r and width Δr and is carried away at the center must be generated outside the radius r. This heat will flow radially through area $2\pi r$ (for unit length) under a gradient $\Delta T/\Delta r$. This gives

$$\Delta T = \frac{q_v(R^2 - r^2)\Delta r}{2kr} \qquad (b)$$

whence

$$\int_{T_1}^{T_2} \Delta T = \frac{q_v}{2k} \int_{r_1}^{r_2} \left(\frac{R^2}{r} - r\right) dr \qquad (c)$$

or

$$T_2 - T_1 = \frac{q_v}{2k}\left[R^2 \ln \frac{r_2}{r_1} - \left(\frac{r_2^2}{2} - \frac{r_1^2}{2}\right)\right] \qquad (d)$$

Using $r_1 = 0.0417$ ft and $r_2 = R = 3.15$ ft (see above), this gives $T_2 - T_1 = 18.1°$F as against 15°F for the simpler calculation above, for the completion of the process in the same time.

The fundamental weakness of both the foregoing calculations is the assumption that the hydration heat is released at a uniform rate and over a period of a month or more. This, in general, is not the case; in fact, most of it may be released in the first few days. We shall accordingly make another calculation, based on (9.8*d*). This will give the temperature at any radius r, t sec after a permanent line source (or sink) has been started in a medium of uniform temperature. This assumes that the medium has been rather quickly raised to this uniform temperature by the release of the heat of hydration and then cools according to the special conditions we shall lay down. While these conditions apparently are not closely related to our problem, we can get some information in this way that will be of interest.

In applying this equation we shall withdraw heat at the same rate as in the two preceding calculations, *viz.*, 41.5 Btu/hr

per ft of pipe. We shall then calculate the temperature of the pipe necessary to do this, at the end of 1, 5, 10, 20, 40, and 60 days. Putting $S' = 41.5/c\rho = 1.22$ and $r = 0.0417$ ft in (9.8*d*), we have, using Appendix F, $T = 16.9°F$ below the initial concrete temperature at the end of the first day of cooling. The values for 5, 10, 20, 40, and 60 days are 20.8, 22.4, 24.1, 25.7, and 26.7°F. The temperatures at radii 1, 2, and 3 ft at the end of 10 days are 7.4, 4.3, and 2.7°F. below the initial temperature, and at the end of 50 days they are 11.2, 7.9, and 6.1°F below this temperature. From these figures we may conclude that a 1-in. pipe held for 52 days at a temperature averaging 25°F below that of a large mass of concrete will withdraw some 51,900 Btu for each foot of pipe length. This is equivalent to the heat of hydration in a cylinder of radius 3.15 ft. During this time the temperature of the immediate surroundings ranges from 2.7°F below the initial value at 3 ft from the pipe after 10 days, to 11.2°F below this value at 1 ft after 50 days—averaging 15 to 20°F above the pipe temperature. These figures are of the order of magnitude encountered in practice.

When these three methods of calculation are compared, the first two assume a uniform rate of heat release and the third a sudden release that raises the mass to its maximum temperature, after which cooling begins. Neither assumption fits the actual case, which lies somewhere between the two. However, all give results of the same order of magnitude, indicating that the largest share of the heat might be withdrawn inside of two months. As a matter of actual practice artificial cooling is usually continued for from 1 to 3 months.

There are two obvious defects in the last solution. The first is the rather trivial one discussed in Sec. 9.10, example 2. The other and more serious one is the fact that it fails to take proper account of the action of neighboring pipes.* In reality each pipe is in effect the center of a cylindrical column of concrete of radius 3.15 ft with no heat transfer across the boundary from one cylinder to another. This and other factors, such as the inevitable rise in cooling water temperature as it flows

* See, however, the suggestions in Sec. 9.11 for treatment of an array of pipes.

through the pipes, are taken into account in the elaborate solution of Kingston.[77]

Some of these same considerations might be utilized in cooling calculations on certain types of uranium (fission) "piles."

9.15. Problems

1. A 50-gm lead ($c = 0.030$; heat of fusion $= 5.47$ cgs) bullet is cast in an iron ($k = 0.144$, $c = 0.105$, $\rho = 7.85$, $\alpha = 0.174$ cgs) mold. Assuming the pouring temperature as 350°C and the mold at zero, find the temperature 3 cm away from the bullet after 10 sec; also find the maximum temperature. Neglect dimensions of the bullet. *Ans.* 2.58°C; 2.64°C

2. If heat equivalent to the combustion of 10^6 kg of coal with a heat of combustion of 7000 cal/gm is suddenly generated at a point in the earth, when will the maximum temperature occur at a point 50 m distant, and what will be its value? Assume $k = 0.0045$, $\alpha = 0.0064$ cgs for the earth concerned. *Ans.* 20.6 years; 5.9°C

3. If the coal of the previous problem burns at a rate of 1,000 kg per day, what will the temperature be at a distance of 10 m from the point in 2 years? [The use of (9.9*b*) should be considered in connection with this and the following problems.] *Ans.* 38°C

4. In a geyser of the type described in Sec. 9.10 make the calculation of the period for $t = 1{,}000$ years. *Ans.* 80 hr

5. In the second or spot-welding example of Sec. 9.13 assume that 200 cal/sec is generated at a point for a period of 10 sec. Calculate the temperature 3 cm from this point at the end of this period. *Ans.* 54°C

6. In the first illustration of Sec. 9.13 assume that the welding machine generates 800 cal/sec per cm of weld for a period of 12 sec. What will be the temperature 3 cm from the weld at the end of this period? *Ans.* 228°C

7. A certain deep mine is to be air-conditioned by the abstraction of 60 Btu/hr for each linear foot of a circular shaft or tunnel 7 ft in diameter. This is driven in rock ($k = 1.2$, $\alpha = 0.032$ fph) initially all at 110°F. What rock-wall temperature might be expected after 10 years of such cooling? *Ans.* 85°F

8. In a heat-pump installation using a 1-in. diameter ground pipe in soil ($k = 1.0$, $\alpha = 0.02$ fph) at a uniform initial temperature of 50°F, heat is withdrawn at an average rate of 10 Btu/hr per linear ft of pipe. What temperature might be expected in the pipe after 2 months of operation? *Ans.* 42°F

CASE III. COOLING OF A SPHERE WITH SURFACE AT CONSTANT TEMPERATURE

9.16. Surface at Zero. To solve this problem we must find a solution of (9.1*c*) that satisfies the boundary conditions

$$T = f(r) \qquad \text{when } t = 0 \tag{a}$$
$$T = 0 \qquad \text{at } r = R \tag{b}$$

Making the substitution $u \equiv rT$ (c)

(9.1c) reduces to

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial r^2} \tag{d}$$

where u must fulfill the conditions

$$u = rf(r) \qquad \text{when } t = 0 \tag{e}$$
$$u = 0 \qquad \text{at } r = R \tag{f}$$
$$u = 0 \qquad \text{at } r = 0 \tag{g}$$

It will be seen that this makes the problem similar to that of the slab (Sec. 8.16) with faces at temperature zero and initial temperature $rf(r)$. With the aid of (8.16*i*) we may then write

$$T = \frac{u}{r} = \frac{2}{Rr} \sum_{m=1}^{\infty} \sin \frac{m\pi r}{R} e^{\frac{-\alpha m^2\pi^2 t}{R^2}} \int_0^R \lambda f(\lambda) \sin \frac{m\pi\lambda}{R}\, d\lambda \tag{h}$$

If the initial temperature is a constant, T_0, we may write (*h*)

$$T = \frac{2T_0}{Rr} \sum_{m=1}^{\infty} \sin \frac{m\pi r}{R} e^{\frac{-\alpha m^2\pi^2 t}{R^2}} \int_0^R \lambda \sin \frac{m\pi\lambda}{R}\, d\lambda \tag{i}$$

But

$$\int_0^R \lambda \sin \frac{m\pi\lambda}{R}\, d\lambda = -\frac{R^2}{m\pi} \cos m\pi \tag{j}$$

so that (*h*) may be written for this case

$$T = \frac{2RT_0}{\pi r}\left(\sin \frac{\pi r}{R} e^{\frac{-\pi^2\alpha t}{R^2}} - \frac{1}{2}\sin\frac{2\pi r}{R} e^{\frac{-4\pi^2\alpha t}{R^2}} + \frac{1}{3}\sin\frac{3\pi r}{R} e^{\frac{-9\pi^2\alpha t}{R^2}} - \cdots\right) \tag{k}$$

Following Sec. 7.14, we may write (*k*) for the case of either heating or cooling, with surface at T_s, as

$$\frac{T - T_s}{T_0 - T_s} = \frac{2R}{\pi r}\left(\sin\frac{\pi r}{R} e^{\frac{-\pi^2\alpha t}{R^2}} - \frac{1}{2}\sin\frac{2\pi r}{R} e^{\frac{-4\pi^2\alpha t}{R^2}} + \cdots\right) \tag{l}$$

9.17. Center Temperature. Equation (9.16*l*) is readily evaluated for the central point if we note that the limit of $(\sin m\pi r/R)/(m\pi r/R) = 1$ as $r \to 0$. Then we have, for a sur-

face temperature T_s,

$$\frac{T_c - T_s}{T_0 - T_s} = 2\,(e^{\frac{-\pi^2\alpha t}{R^2}} - e^{\frac{-4\pi^2\alpha t}{R^2}} + e^{\frac{-9\pi^2\alpha t}{R^2}} - \cdots) \equiv B(x) \qquad (a)$$

where $x = \pi^2\alpha t/R^2$. $B(x)$ is tabulated in Appendix H.

9.18. Average Temperature. The average temperature T_a of the sphere at any time t may be found from (9.16k) by multiplying each element of volume by its corresponding temperature, summing such terms for the whole sphere, and dividing by the volume of the sphere. Thus, since T is a function of r,

$$T_a = \frac{3}{4\pi R^3}\int_0^R T4\pi r^2\,dr \qquad (a)$$

$$= \frac{6T_0}{\pi R^2}\left(e^{\frac{-\pi^2\alpha t}{R^2}}\int_0^R r\sin\frac{\pi r}{R}\,dr - \frac{1}{2}e^{\frac{-4\pi^2\alpha t}{R^2}}\int_0^R r\sin\frac{2\pi r}{R}\,dr + \frac{1}{3}e^{\frac{-9\pi^2\alpha t}{R^2}}\int_0^R r\sin\frac{3\pi r}{R}\,dr - \cdots\right) \qquad (b)$$

$$= \frac{6T_0}{\pi^2}\left(e^{\frac{-\pi^2\alpha t}{R^2}} + \frac{1}{4}e^{\frac{-4\pi^2\alpha t}{R^2}} + \frac{1}{9}e^{\frac{-9\pi^2\alpha t}{R^2}} + \cdots\right) \qquad (c)$$

or, in general,

$$\frac{T_a - T_s}{T_0 - T_s} = \frac{6}{\pi^2}\left(e^{\frac{-\pi^2\alpha t}{R^2}} + \frac{1}{4}e^{\frac{-4\pi^2\alpha t}{R^2}} + \cdots\right) \equiv B_a(x)^* \qquad (d)$$

where $x = \pi^2\alpha t/R^2$.

APPLICATIONS

9.19. Mercury Thermometer. Equations (9.18c) and (9.18d) may be applied to a spherical-bulb thermometer immersed in a stirred liquid. Neglecting the effect of the glass shell, the temperature of the mercury is given to a close approximation by the first term of the equation unless t is very small. The rate of cooling is then

$$-\frac{\partial T_a}{\partial t} = \frac{\pi^2\alpha(T_a - T_s)}{R^2} \qquad (a)$$

9.20. Spherical Safes. Compare the fire-protecting qualities of two safes of solid steel ($\alpha = 0.121$ cgs) and solid concrete

* See Appendix H.

($\alpha = 0.0058$ cgs), each spherical in form, of diameter 150 cm (59 in.) and of very small internal cavity. Assuming that the surfaces are quickly raised from initial temperatures of 20°C (68°F) to 500°C (932°F), determine the temperatures at the centers after various times.

Using (9.17*a*) and Appendix H, we find that the temperature in the center of the steel safe would be 98°C (208°F) at the end of 1 hr and 455°C (850°F) after 4 hr, while in concrete the temperatures would run only 25°C (77°F) at the end of 10 hr and not exceed 130°C (266°F) before 24 hr. Obviously, this comparison is hardly fair to the steel safe since it would be practically impossible to raise its surface temperature as rapidly as is assumed here.

9.21. Steel Shot. Such a shot or ball 3 cm (1.18 in.) in diameter, at 800°C (1472°F), has its surface suddenly chilled to 20°C (68°F); what is the temperature 1 cm below the surface in 1.8 sec? Putting $r = 0.5$ and $R = 1.5$, also $\alpha = 0.121$ in (9.16*l*), we readily find T to be 501°C (934°F). It will be noted that the cooling is much more rapid than in the case treated in Sec. 7.22.

The rate of cooling may be found by differentiating (9.16*l*) with respect to t. This gives

$$\frac{\partial T}{\partial t} = -2\frac{\pi\alpha(T_0 - T_s)}{Rr}\left(\sin\frac{\pi r}{R}e^{\frac{-\pi^2\alpha t}{R^2}} - 2\sin\frac{2\pi r}{R}e^{\frac{-4\pi^2\alpha t}{R^2}} + \cdots\right) \qquad (a)$$

This equation might be used in an investigation of the relation between rapidity of cooling and hardness for approximately spherical steel ingots. The preceding equations might also be applied to a large number of practical problems of somewhat the same nature as those discussed in previous chapters, by treating all roughly spherical shapes as spheres. The theory might prove of service in such problems as the annealing of large steel castings or in a study of the temperature stresses and consequent tendency to cracking that accompanies the quenching of large steel ingots.

9.22. Household Applications. There are numerous everyday examples of the type of heat-conduction problem discussed in these sections. The processes of roasting meats, boiling

potatoes or eggs, cooling of melons, etc., all involve the heating or cooling of roughly spherical bodies under conditions of reasonably constant surface temperature. As an example, we may question how long a spherical potato 7 cm in diameter must be in boiling water before the center attains a temperature of 90°C, assuming an initial temperature of 20°C. We may use the same diffusivity as for water ($\alpha = 0.00143$ cgs) for this and other vegetables and fruits. Then, using (9.17*a*), we have $90 - 100 = (20 - 100)B(x)$, which, from Appendix H, gives $x(= \pi^2\alpha t/3.5^2) = 2.76$, or $t = 2{,}400$ sec or 40 min. It may be remarked that unless the potato is in rapidly boiling, *i.e.*, vigorously stirred, water, the surface will not attain the 100°C rapidly and the cooking process will accordingly take longer.

Tradition requires that ivory billiard balls, after exposure to violent temperature change, should be allowed to remain in constant temperature surroundings for a matter of several hours before being used for play. For such a ball 6.35 cm (2.5 in.) in diameter we may inquire how long it will be before the center temperature change is 99 per cent of the surface change. Using $\alpha = 0.002$ cgs, we have from (9.17*a*), $1 = 100\ B(x)$, or, from Appendix H, $x = 5.3$. This means that the temperature should be uniform throughout to within 1 per cent in 2,710 sec, or considerably less than 1 hr. It would seem then that this tradition must be explained on a basis of other than temperature considerations alone.

9.23. Problems

1. The surface of a sphere of cinder concrete ($\alpha = 0.0031$ cgs) 30 cm in diameter is rapidly raised to 1500°C and held there. If it is all initially at zero, what will be the temperature of the center in 1 hr? In 5 hr?

Ans. 49°C; 1240°C

2. A mercury thermometer, with a spherical bulb 1 cm in diameter, at 40°C is immersed in a stirred mixture of ice and water. Neglecting the glass envelope and assuming that the surface is instantly chilled to zero, determine how soon the average temperature is within 0.01°C of the bath. Use $\alpha = 0.044$ cgs.

Ans. 4.5 sec

3. An egg equivalent to a sphere 4.4 cm in diameter and at 20°C is placed in boiling water. Calculate the center and also average temperatures in 3 min. Solve the same problem for a 30-cm diameter melon at 20°C in ice water for 3 hr; for 6 hr. Assume $\alpha = 0.00143$ cgs in each case.

Ans. Egg, 23.6 and 69.7°C; melon, center, 17.7 and 10.2°C, and average, 6.4 and 3.2°C

4. Show that the common rule for roasting meats—of allowing so much time per pound but decreasing somewhat this allowance per pound for the larger roasts—rests on a good theoretical basis.

CASE IV. THE COOLING OF A SPHERE BY RADIATION

9.24. We shall now solve a more difficult problem than any we have before attempted, *viz.*, that of the temperature state in a sphere cooled by radiation. The solution will apply to the case of the sphere either in air or *in vacuo*, for the only assumption made in regard to the loss of heat is that Newton's law of cooling holds; *i.e.*, that the rate of loss of heat by a surface is proportional to the difference between its temperature and that of the surroundings. This does not hold for large temperature differences. See Sec. 2.5.

As we shall see, the solution can also be applied to the case of a sphere of metal or other material of high conductivity, covered with a thin coating of some poorly conducting substance and placed in a bath at constant temperature. For the rate of loss of heat by the surface of the metal sphere will be proportional to the temperature gradient through the surface coating, *i.e.*, to the difference of temperature between the inner and outer surfaces of this coating, which, by the conditions of the problem, is equal to the difference of temperature of the metal surface and the bath. An example of this latter case is the mercury thermometer with a spherical bulb, immersed in a liquid, it being desired to make correction for the glass envelope.

9.25. The differential equation for this case is, as before,

$$\frac{\partial(rT)}{\partial t} = \alpha \frac{\partial^2(rT)}{\partial r^2} \qquad (a)$$

with the boundary conditions

$$T = f(r) \qquad \text{when } t = 0 \qquad (b)$$

$$-k\frac{\partial T}{\partial r} = hT \qquad \text{at } r = R \qquad (c)$$

The last condition states that the rate at which heat is brought to unit area of the surface by conduction, *viz.*, $-k(\partial T/\partial r)$, must

be the rate at which it is radiated from this area, and this is hT, where h is the emissivity of the surface. The surroundings are supposed to be at zero.

As before, put
$$u \equiv rT \tag{d}$$
Then we have
$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial r^2} \tag{e}$$
and the conditions
$$u = rf(r) \qquad \text{when } t = 0 \tag{f}$$
$$u = 0 \qquad \text{at } r = 0 \tag{g}$$
$$\frac{\partial u}{\partial r} + \left(C - \frac{1}{R}\right) u = 0 \qquad \text{at } r = R \tag{h}$$

where, for short, C is written for h/k.

Now we have already seen in Sec. 7.2 that
$$u = e^{-\alpha m^2 t} \cos mr \tag{i}$$
and
$$u = e^{-\alpha m^2 t} \sin mr \tag{j}$$

are particular solutions of (e). Solution (i) is excluded by condition (g), but (j) satisfies this condition for all values of m. To see if (h) is also fulfilled, we substitute the value of u from (j) and get
$$mR \cos mR = (1 - CR) \sin mR \tag{k}$$

If m_p is a root of this transcendental equation, then
$$u = e^{-\alpha m_p^2 t} \sin m_p r \tag{l}$$

is a particular solution of (e) satisfying (g) and (h). We must now endeavor to build up, with the aid of terms of the type (l), a solution that will also satisfy (f).

Since the sum of a number of particular solutions of a linear, homogeneous partial differential equation is also a solution, we note that
$$u = B_1 e^{-\alpha m_1^2 t} \sin m_1 r + B_2 e^{-\alpha m_2^2 t} \sin m_2 r + B_3 e^{-\alpha m_3^2 t} \sin m_3 r + \cdots \tag{m}$$

where B_1, B_2, B_3, . . . are arbitrary constants, is a solution of (e) satisfying (g). It moreover satisfies (h) if m_1, m_2, m_3, . . . are roots of (k). It evidently reduces for $t = 0$ to
$$B_1 \sin m_1 r + B_2 \sin m_2 r + B_3 \sin m_3 r + \cdots \tag{n}$$

and if it is possible to develop $rf(r)$, for all values of r between 0 and R, in terms of such a series, we shall have (f) satisfied as well.

9.26. The solution of our problem, then, will consist of two parts: (1) the solution of the transcendental equation (9.25k), *i.e.*, the determination of the roots m_1, m_2, m_3, . . . (we anticipate a fact shortly to be shown, *viz.*, that there are an infinite number of such roots); and (2) the expansion of the function $rf(r)$ in the sine series (9.25n). The second part of the problem is analogous to development in terms of a Fourier's series, but more complicated because the numbers m_1, m_2, m_3, instead of being the integers 1, 2, 3, as in the regular Fourier's series, must in the present case be roots of equation (9.25k).*

9.27. The Solution of the Transcendental Equation. The roots of (9.25k) are easily obtained by computation, but a study of their values under various conditions may be most easily made by graphical methods. If we make the substitutions

$$\gamma \equiv mR \tag{a}$$

and

$$\beta \equiv 1 - CR \tag{b}$$

(9.25k) becomes

$$\gamma \cos \gamma = \beta \sin \gamma \tag{c}$$

or, more simply,

$$\gamma = \beta \tan \gamma \tag{d}$$

Then, if we construct the curves

$$y = \tan x \tag{e}$$

and

$$y = \frac{x}{\beta} \tag{f}$$

their points of intersection will give the values of x for which

$$\frac{x}{\beta} = \tan x \tag{g}$$

i.e., the roots of (d) and hence of (9.25k).

9.28. We may draw some general conclusions as to these roots. In the first place, there are evidently an infinite number of positive roots, and the same number of negative, which are

* This is the most general sine development that can be obtained by Fourier's method. See Byerly.[23, p. 121]

equal in absolute value to the positive. The values of the roots vary between certain limits with the slope of the line $y = x/\beta$, *i.e.*, with the value of C, or h/k. Since C can have, theoretically at least, any value between 0 and ∞ but must always be positive, the slope

$$\frac{1}{\beta} = \frac{1}{1 - CR} \qquad (a)$$

can have any value between 1 and ∞ or between 0 and $-\infty$.

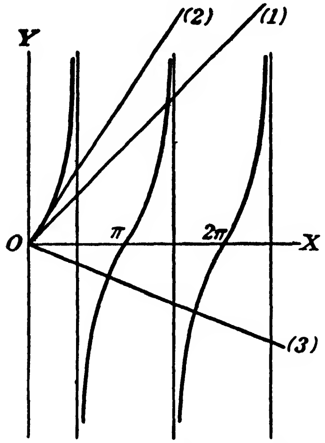

FIG. 9.2. Curves whose intersections give the roots for Sec. 9.28.

We can easily show with the aid of a figure the approximate values of the roots for the several cases as follows:

Let $C = 0$, corresponding to the case of a sphere protected with a thermally impervious covering. The roots then correspond to the intersections of the line (1) (Fig. 9.2) of 45 deg slope. Their values are $0, \gamma_1, \gamma_2, \ldots$, where

$$\pi < \gamma_1 < \frac{3\pi}{2}; \qquad 2\pi < \gamma_2 < \frac{5\pi}{2}; \qquad \ldots$$

$$n\pi < \gamma_n < \left(n + \frac{1}{2}\right)\pi \qquad (b)$$

γ_n in this case approaches the limit $(n + \frac{1}{2})\pi$ as n increases.

Next, let C lie between 0 and $1/R$ so that $0 < (1 - CR) < 1$. The line (2) corresponds to this case, and the roots $0, \gamma_1, \gamma_2, \gamma_3, \ldots$ have the values

$$0 < \gamma_1 < \frac{\pi}{2}; \qquad \pi < \gamma_2 < \frac{3\pi}{2}; \qquad \ldots$$

$$(n - 1)\pi < \gamma_n < \left(n - \frac{1}{2}\right)\pi \qquad (c)$$

approaching the larger values as C increases. When $C = 1/R$, then the roots become

$$0, \quad \frac{\pi}{2}, \quad \frac{3\pi}{2}, \quad \frac{5\pi}{2}, \ldots \qquad (d)$$

Finally, if C lies between $1/R$ and ∞, the intersecting straight line will fall below the axis in some position such as (3), and the roots $0, \gamma_1, \gamma_2, \ldots$ will have values

$$\frac{\pi}{2} < \gamma_1 < \pi; \qquad \frac{3\pi}{2} < \gamma_2 < 2\pi; \qquad \cdots$$
$$\left(n - \frac{1}{2}\right)\pi < \gamma_n < n\pi \cdots \tag{e}$$

which become for $C = \infty$

$$\gamma_1 = \pi, \qquad \gamma_2 = 2\pi, \qquad \cdots, \qquad \gamma_n = n\pi \cdots \tag{f}$$

From these roots $\gamma_1, \gamma_2, \gamma_3$, the values $m_1, m_2, m_3, \ldots$ satisfying (9.25k) are obtained at once with the aid of (9.27a).

9.29. The General Sine Series Development. We shall arrive at this development by assuming that it is possible to expand $rf(r)$ in a series

$$rf(r) = B_1 \sin m_1 r + B_2 \sin m_2 r + \cdots$$
$$+ B_b \sin m_b r + \cdots \equiv \sum_{b=1}^{\infty} B_b \sin m_b r \tag{a}$$

just as we assumed before that such a function could be expanded in an ordinary Fourier's series, and then proceed to find the values of the coefficients $B_1, B_2, B_3, \ldots$, to which this assumption leads. The values $m_1, m_2, m_3, \ldots$ are the roots of equation (9.25k) determined above. While zero is a root in each case, there is no corresponding term in the series since $\sin 0 = 0$. The negative roots that occur are included with the positive in the terms of (a), for since $\sin(-x) = -\sin x$, we may write

$$B_b' \sin m_b r + B_b'' \sin(-m_b r) = B_b \sin m_b r \tag{b}$$

Multiplying each side of (a) by $\sin m_a r\, dr$ and integrating from 0 to R,

$$\int_0^R rf(r) \sin m_a r\, dr = \sum_{b=1}^{\infty} B_b \int_0^R \sin m_b r \sin m_a r\, dr \tag{c}$$

Now $\displaystyle\int_0^R \sin m_b r \sin m_a r\, dr$

$$= \frac{1}{2}\int_0^R \{\cos[(m_b - m_a)r] - \cos[(m_b + m_a)r]\}\, dr \tag{d}$$

$$= \frac{\sin [(m_b - m_a)R]}{2(m_b - m_a)} - \frac{\sin [(m_b + m_a)R]}{2(m_b + m_a)} \tag{e}$$

$$= \frac{(m_a \sin m_bR \cos m_aR - m_b \cos m_bR \sin m_aR)}{(m_b^2 - m_a^2)} \tag{f}$$

But since m_a and m_b are roots of (9.25k),

$$m_aR = (1 - CR) \tan m_aR; \qquad m_bR = (1 - CR) \tan m_bR \tag{g}$$

so that
$$m_a \tan m_bR = m_b \tan m_aR \tag{h}$$
or
$$m_a \sin m_bR \cos m_aR = m_b \sin m_aR \cos m_bR \tag{i}$$

Therefore,
$$\int_0^R \sin m_br \sin m_ar\, dr = 0 \tag{j}$$

when m_a and m_b are different. If they are equal, we have

$$\int_0^R \sin^2 m_ar\, dr = \frac{1}{2} \int_0^R (1 - \cos 2m_ar)\, dr \tag{k}$$

$$= \frac{R}{2} - \frac{\sin 2m_aR}{4m_a} \tag{l}$$

Now
$$\sin 2m_aR = \frac{2 \tan m_aR}{1 + \tan^2 m_aR} \tag{m}$$

$$= \frac{2m_aR(1 - CR)}{(CR - 1)^2 + m_a^2R^2} \tag{n}$$

Therefore,
$$\int_0^R \sin^2 m_ar\, dr = \frac{R}{2} \frac{m_a^2R^2 + CR(CR - 1)}{m_a^2R^2 + (CR - 1)^2} \tag{o}$$

Applying this in the series (c), *i.e.*, in

$$\int_0^R rf(r) \sin m_ar\, dr = B_1 \int_0^R \sin m_1r \sin m_ar\, dr + B_2 \int_0^R \sin m_2r \sin m_ar\, dr + \cdots \tag{p}$$

we have
$$B_a = \frac{2}{R} \frac{m_a^2R^2 + (CR - 1)^2}{m_a^2R^2 + CR(CR - 1)} \int_0^R rf(r) \sin m_ar\, dr \tag{q}$$

9.30. Final Solution. Our problem is now solved, for we have evaluated the coefficients of the series (9.29a) in terms of the roots of equation (9.25k), which roots we have shown to have real values that are easily determined. The solution may be written

$$u = \sum_{a=1}^{\infty} B_a e^{-\alpha m_a^2 t} \sin m_ar \tag{a}$$

or, evaluating B_a from (9.29q) and remembering that $u = rT$,

$$T = \frac{2}{rR} \sum_{a=1}^{\infty} \frac{m_a^2R^2 + (CR - 1)^2}{m_a^2R^2 + CR(CR - 1)} e^{-\alpha m_a^2 t} \sin m_a r \cdot \int_0^R \lambda f(\lambda) \sin m_a \lambda \, d\lambda \quad (b)$$

9.31. Initial Temperature T_0. In the case in which the initial temperature of the sphere is everywhere the same, *i.e.*, $f(r) = T_0$, we find that the above integral reduces to

$$T_0 \int_0^R \lambda \sin m\lambda \, d\lambda = \frac{T_0}{m^2} (\sin mR - mR \cos mR) \quad (a)$$

and, with the use of (9.25k),
$$= \frac{CRT_0}{m^2} \sin mR \quad (b)$$

Thus, (9.30b) becomes for this case

$$T = \frac{2CT_0}{r} \left\{ \frac{m_1^2R^2 + (CR - 1)^2}{m_1^2[m_1^2R^2 + CR(CR - 1)]} e^{-\alpha m_1^2 t} \sin m_1 R \sin m_1 r + \frac{m_2^2R^2 + (CR - 1)^2}{m_2^2[m_2^2R^2 + CR(CR - 1)]} e^{-\alpha m_2^2 t} \sin m_2 R \sin m_2 r + \cdots \right\} \quad (c)$$

9.32. Special Cases. If CR is small in comparison with unity, as it would be in many cases, the problem is greatly simplified. For an inspection of Fig. 9.2 shows that in this case m_1R will be very small, while the other values of mR will be larger than π, so that only the first term of the series (9.31c) need be considered. The value of m_1 is readily determined from (9.25k) by developing the sine and cosine in series and neglecting higher powers of m_1R, in which case we obtain

$$1 - \tfrac{1}{2}m_1^2R^2 = (1 - CR)(1 - \tfrac{1}{6}m_1^2R^2) \quad (a)$$

from which it follows that

$$m_1^2 = \frac{3C}{R} \quad (b)$$

With the aid of (b), equation (9.31c) may be still further simplified if it be remembered that m_1R and m_1r are small quan-

tities, and if C^2R^2 is neglected, for it reduces at once to

$$T = T_0 e^{-3Cat/R} \tag{c}$$
$$= T_0 e^{-3ht/c\rho R} \tag{d}$$

c being the specific heat.

9.33. The assumptions involved in this last formula are that the sphere is so small or the cooling so slow that the temperature at any time is sensibly uniform throughout the whole volume. With this assumption it may be derived independently in a very simple manner; for the quantity of heat that the sphere radiates in time dt is

$$4\pi R^2 hT\,dt \tag{a}$$

This means a change in temperature of the sphere of dT, which corresponds to a quantity of heat given up equal to

$$-\tfrac{4}{3}\pi R^3 c\rho\,dT \tag{b}$$

the negative sign being used, since dT is a negative quantity. Hence, we have

$$4\pi R^2 hT\,dt = -\tfrac{4}{3}\pi R^3 c\rho\,dT \tag{c}$$

the integration of which gives, since the temperature of the sphere is T_0 at the time $t = 0$,

$$T = T_0 e^{-3ht/c\rho R} \tag{d}$$

as above.

9.34. Applications. Equations (9.30b) and (9.31c) make possible the treatment of the problem of the cooling of the earth by radiation* before the formation of a surface crust, which was kept, by the evaporation of the water, at a nearly constant temperature. The solutions of Cases III and IV of the present chapter would enable one to treat the problem of terrestrial temperatures with account taken of the spherical shape of the earth, but as already noted our present data would by no means warrant such a rigorous solution, which would alter the result in any case by only a very small fraction. It may be noted that the solution of the problem of radiation for the semiinfinite

* However, see Sec. 2.5 in this connection.

solid is gained from the present case by letting R approach infinity.

As already suggested, the solution for the present case will fit another that at first sight seems quite foreign to it, *viz.*, the cooling of a mercury-in-glass thermometer in a liquid. If the glass is so thin, as it usually is, that its heat capacity can be neglected, we have only to set in place of h, in the above equations, k/l, where l is the thickness of the glass and k its conductivity, and we shall have a solution of this problem.

The general case of cooling or heating roughly spherical bodies by convection or radiation—especially in its simpler phases—has many applications. Most of these, however, are beyond the scope of this book since conduction in many of them plays a secondary part. Students who are interested in pursuing the general subject of heat transfer may profitably consult Brown and Marco,[20] Croft,[34] Gröber,[53] Jakob and Hawkins,[68] McAdams,[90] Schack,[122] Stoever,[139,140] Vilbrandt, [156] and similar books.

9.35. Problems

1. A wrought-iron cannon ball of 10 cm radius and at a uniform temperature of 50°C is allowed to cool by radiation in a vacuum to surroundings at 30°C. If the value of h for the surface is 0.00015 cal/(sec)(cm²)(°C), what will be the temperature at the center and at the surface after 1 hr? Use $k = 0.144$, $\alpha = 0.173$ cgs, for iron. *Ans.* 46.5°, 46.4°C

2. A thermometer with spherical mercury bulb of 3.5 mm outside and 2.5 mm inside radius, heated to an initial temperature of 30°C, is plunged into stirred ice water. Find, to a first approximation, how long it will be before the temperature at its center will fall to within ¼°C of that of the bath. Neglect the heat capacity but not the conductivity of the glass (use $k = 0.0024$ cgs). For mercury use $c = 0.033$, $\rho = 13.6$ cgs. *Ans.* 7.5 sec

3. The initial temperature of an orange 10 cm in diameter is 15°C while the surroundings are at 0°C. If the emissivity of the surface is 0.00025 cgs and the thermal constants of the orange the same as those of water, what will be the temperature 1 cm below the surface after 8 hr? *Ans.* 0.38°C

CASE V. FLOW OF HEAT IN AN INFINITE CIRCULAR CYLINDER

9.36. Bessel Functions. In order to solve the problem of the unsteady state in the cylinder we must gain a slight acquaint-

ance with some of the simpler properties of Bessel functions.* The function $J_0(z)$ defined by the series

$$J_0(z) \equiv 1 - \frac{z^2}{2^2} + \frac{z^4}{2^2 \cdot 4^2} - \frac{z^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots \qquad (a)$$

is called a "Bessel function of order zero." If n is zero or a positive integer, $J_n(z)$, of order n, is defined by the series

$$J_n(z) \equiv \frac{z^n}{2^n \cdot n!}\left[1 - \frac{z^2}{2(2n+2)} + \frac{z^4}{2 \cdot 4(2n+2)(2n+4)} - \frac{z^6}{2 \cdot 4 \cdot 6(2n+2)(2n+4)(2n+6)} + \cdots\right] \qquad (b)$$

Putting $0! = 1$ (*i.e.*, $1!/1$), the above is seen to reduce to (a) for $n = 0$. If we write $J_0'(z)$ for the derivative $dJ_0(z)/dz$, it is seen at once that

$$J_0'(z) = -J_1(z)\dagger \qquad (c)$$

It can also be shown that

$$\frac{d}{dz}[z^n J_n(z)] = z^n J_{n-1}(z) \qquad (d)$$

9.37. From an inspection of (4.6a) we can write at once for the Fourier equation in cylindrical coordinates, if T is a function of r and t only,

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right) \qquad (a)$$

We shall use this in solving the problem of the nonsteady state in a long cylinder of radius R under conditions of purely radial flow.

9.38. Surface at Zero. To solve this problem we must find a solution of (9.37a) that satisfies the boundary conditions

$$T = f(r) \quad \text{when } t = 0, (r \leqq R) \qquad (a)$$
$$T = 0 \quad \text{at } r = R \qquad (b)$$

Making the substitution $T \equiv ue^{-\alpha\beta^2 t}$ (c)

where u is a function of r only and β a number whose value will

* See, *e.g.*, Watson,[159] Carslaw,[27] McLachlan.[93]

† Tables of $J_0(z)$ and $J_1(z)$ are given in Appendix I.

be investigated later, (9.37a) becomes

$$\frac{\partial T}{\partial t} = ue^{-\alpha\beta^2 t}(-\alpha\beta^2) = \alpha\left(e^{-\alpha\beta^2 t}\frac{d^2u}{dr^2} + \frac{1}{r}e^{-\alpha\beta^2 t}\frac{du}{dr}\right) \tag{d}$$

or

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} + \beta^2 u = 0^* \tag{e}$$

which is known as a "Bessel equation of order zero." Now, as is easily shown by differentiation, $u = J_0(\beta r)$ is a solution of (e). Thus,

$$T = BJ_0(\beta r)e^{-\alpha\beta^2 t} \tag{f}$$

is a particular solution of (9.37a) suitable for our problem.

To satisfy condition (b) we must have

$$J_0(\beta R) = 0 \tag{g}$$

The values of β_1, β_2, . . . that satisfy this equation for any particular value of R may be obtained from Appendix I. If $f(r)$ can be expanded in the series

$$f(r) = B_1 J_0(\beta_1 r) + B_2 J_0(\beta_2 r) + \cdots \tag{h}$$

condition (a) will also be satisfied and the solution of the problem will be

$$T = \sum_{m=1}^{\infty} B_m J_0(\beta_m r)e^{-\alpha\beta_m^2 t} \tag{i}$$

In evaluating B_1, B_2, . . . we follow a procedure not unlike that employed in Sec. 6.2 in determining the Fourier coefficients. Multiply both sides of (h) by $rJ_0(\beta_m r)\,dr$ and integrate from 0 to R. Then,

$$\int_0^R rf(r)J_0(\beta_m r)\,dr = B_1\int_0^R rJ_0(\beta_1 r)J_0(\beta_m r)\,dr + \cdots + B_m\int_0^R r[J_0(\beta_m r)]^2\,dr + \cdots \tag{j}$$

Now it can be shown† that

$$\int_0^R rJ_0(\beta_m r)J_0(\beta_p r)\,dr = 0 \tag{k}$$

* This is commonly written

$$\frac{d^2u}{d(\beta r)^2} + \frac{1}{\beta r}\frac{du}{d(\beta r)} + u = 0$$

† Carslaw.[27, pp. 116,117]

and also

$$\int_0^R r[J_0(\beta_m r)]^2\,dr = \frac{R^2}{2}\,[J_0'(\beta_m R)]^2 \tag{l}$$

Then, substituting from (9.36c) for J_0', we have

$$B_m = \frac{2}{R^2}\,\frac{\int_0^R rf(r)J_0(\beta_m r)\,dr}{[J_1(\beta_m R)]^2} \tag{m}$$

Therefore, the final solution is

$$T = \frac{2}{R^2}\sum_{m=1}^{\infty} e^{-\alpha\beta_m{}^2t}\,\frac{\int_0^R rf(r)J_0(\beta_m r)\,dr}{[J_1(\beta_m R)]^2}\,J_0(\beta_m r) \tag{n}$$

When $f(r) = T_0$, a constant, we evaluate (n) as follows:

$$T_0\int_0^R rJ_0(\beta_m r)\,dr = \frac{T_0}{\beta_m^2}\int_0^{\beta_m R}(\beta_m r)J_0(\beta_m r)\,d(\beta_m r) \tag{o}$$

and from (9.36d) this equals

$$\frac{T_0}{\beta_m^2}\,(\beta_m r)J_1(\beta_m r)\Big|_0^{\beta_m R} = \frac{T_0 R}{\beta_m}\,J_1(\beta_m R) \tag{p}$$

which means that (n) reduces to

$$T = 2T_0\sum_{m=1}^{\infty} e^{-\alpha\beta_m{}^2t}\,\frac{J_0(\beta_m r)}{\beta_m R J_1(\beta_m R)} \tag{q}$$

A more easily usable form is obtained by writing

$$\beta_m \equiv \frac{z_m}{R} \tag{r}$$

where z_m is the mth root of $J_0(z) = 0$.

Thus, we have finally, for a body at T_0 and surface at T_s,

$$\frac{T - T_s}{T_0 - T_s} = 2\sum_{m=1}^{\infty}\frac{J_0(z_m r/R)}{z_m J_1(z_m)}\,e^{-\alpha t z_m{}^2/R^2} \tag{s}$$

which holds for either heating or cooling.

If we are interested only in the temperature T_c at the center

where $r = 0$, (s) becomes

$$\frac{T_c - T_s}{T_0 - T_s} = 2 \sum_{m=1}^{\infty} \frac{e^{-\alpha t z_m^2/R^2}}{z_m J_1(z_m)} \equiv C(x) \tag{t}$$

where $x = \alpha t/R^2$. Values of this series are tabulated in Appendix J.

APPLICATIONS

9.39. Timbers; Concrete Columns. MacLean[95] has made extensive studies of the heating of various woods, using equations like the preceding in connection with round timbers. Computations of center temperatures may be very easily made with the aid of Appendix J. As an example, let us calculate the temperature at the center (and not near the ends) of a round oak ($\alpha = 0.0063$ fph) log 12 in. in diameter, 8 hr after it has been placed in a steam bath. Initial temperature is 60°F and steam temperature 260°F. Using (9.38t) and putting $x = \alpha t/R^2 = 0.201$, we have from Appendix J, $C(0.201) = 0.498$, and therefore $T = 161$°F.

For points not on the axis the calculations are not so simple. As an example, suppose that a long circular column of concrete ($\alpha = 0.03$ fph) 3 ft in diameter and initially at 50°F has its surface suddenly heated to 450°F. What will be the temperature at a depth of 6 in. below the surface after 2 hr?

We use (9.38s). The values of z to satisfy (9.38g), *i.e.*, $J_0(z) = 0$, are found from Appendix I, Table I.2, to be $z_1 = 2.405$; $z_2 = 5.520$; $z_3 = 8.654$; $z_4 = 11.79$. Using Table I.1 of Appendix I, we find that the corresponding values for $J_0(z_m r/R)$ are 0.454, −0.398, 0.082, and 0.203; and for $J_1(z_m)$, 0.519, −0.340, 0.271, and −0.232. Putting these values in the various terms of the series, we finally get $T = 123$°F.

Problems of this type are important in connection with fireproofing considerations when it is important to know how long it will take supporting columns to get dangerously hot in a fire.

9.40. Problems

1. In the second application of Sec. 9.39 calculate the temperature after 4 hr at a depth of 6 in. below the surface and also at the center.

Ans. 202°F; 57°F

2. A long glass rod ($\alpha = 0.006$ cgs) of radius 5 cm and at 100°C has its surface suddenly cooled to 20°C. What is the temperature at the center after 8 min? *Ans.* 83.3°C

CASE VI. GENERAL CASE OF HEAT FLOW IN AN INFINITE MEDIUM

9.41. In Case II of this chapter we solved the problem of the flow of heat from an instantaneous point source. We shall extend this result to cover the case in which we have an initial arbitrary distribution of heat, the initial temperature being given as a function of the coordinates in three dimensions.

Let x,y, and z be the coordinates of any point whose temperature we wish to investigate at any time t, while λ,μ,ν are the coordinates of any heated element of volume and become in general the variables of integration. Then, the initial temperature is

$$T_0 = f(\lambda,\mu,\nu) \tag{a}$$

and the quantity of heat initially contained in any volume element $d\lambda\, d\mu\, d\nu$ is

$$dQ = f(\lambda,\mu,\nu)\frac{k}{\alpha}\, d\lambda\, d\mu\, d\nu \tag{b}$$

If this quantity of heat is propagated through the body, it will produce a rise in temperature which can be obtained at once from (9.5i), and which is, since

$$r^2 = (\lambda - x)^2 + (\mu - y)^2 + (\nu - z)^2 \tag{c}$$

$$dT = \left(\frac{\eta}{\sqrt{\pi}}\right)^3 e^{-[(\lambda-x)^2+(\mu-y)^2+(\nu-z)^2]\eta^2} f(\lambda,\mu,\nu)\, d\lambda\, d\mu\, d\nu \tag{d}$$

The temperature at any point will be the sum of all these increments of temperature and may be obtained by integrating (d):

$$T = \left(\frac{\eta}{\sqrt{\pi}}\right)^3 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-[(\lambda-x)^2+(\mu-y)^2+(\nu-z)^2]\eta^2} f(\lambda,\mu,\nu)\, d\lambda\, d\mu\, d\nu \tag{e}$$

Making the substitutions

$$\beta \equiv (\lambda - x)\eta; \qquad \gamma \equiv (\mu - y)\eta; \qquad \epsilon \equiv (\nu - z)\eta \tag{f}$$

this becomes

$$T = \frac{1}{\pi^{3/2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\beta^2-\gamma^2-\epsilon^2} f\left(x + \frac{\beta}{\eta}, y + \frac{\gamma}{\eta}, z + \frac{\epsilon}{\eta}\right) d\beta\, d\gamma\, d\epsilon \qquad (g)$$

9.42. It will be instructive to show how this solution may be obtained independently as a particular integral of the conduction equation

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) \qquad (a)$$

subject to the boundary condition

$$T = f(\lambda,\mu,\nu) \qquad \text{when } t = 0 \qquad (b)$$

Assume $T = XYZ$, where X is a function of x and t, and where Y and Z are functions of y,t and z,t, respectively. Then we have from (a)

$$YZ\frac{\partial X}{\partial t} + XZ\frac{\partial Y}{\partial t} + XY\frac{\partial Z}{\partial t}$$
$$= \alpha\left(YZ\frac{\partial^2 X}{\partial x^2} + XZ\frac{\partial^2 Y}{\partial y^2} + XY\frac{\partial^2 Z}{\partial z^2}\right) \qquad (c)$$

But since X,Y, and Z are essentially independent, being functions of the independent variables x,y,z, this can only be true if the corresponding terms on each side of the equation are equal, *i.e.*, if

$$\frac{\partial X}{\partial t} = \alpha\frac{\partial^2 X}{\partial x^2} \qquad (d)$$

with similar equations for Y and Z.

Now it may be easily shown by differentiation that

$$X = \frac{1}{\sqrt{t}} e^{-(\lambda-x)^2\eta^2} \qquad (e)$$

is a particular solution of (d), a type of solution already made use of in Sec. 8.3, so that

$$T = \frac{1}{\sqrt{t}} e^{-(\lambda-x)^2\eta^2} \frac{1}{\sqrt{t}} e^{-(\mu-y)^2\eta^2} \frac{1}{\sqrt{t}} e^{-(\nu-z)^2\eta^2} \qquad (f)$$

is a solution of (a). Therefore, if C is any constant, and $\Psi(\lambda,\mu,\nu)$ an arbitrary function of λ,μ,ν,

$$T = C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{t}}\right)^3 e^{-[(\lambda-x)^2+(\mu-y)^2+(\nu-z)^2]\eta^2} \cdot \Psi(\lambda,\mu,\nu)\,d\lambda\,d\mu\,d\nu \qquad (g)$$

is also a solution of (a). By the substitutions (9.41f) this reduces to

$$T = C(2\sqrt{\alpha})^3\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\beta^2-\gamma^2-\epsilon^2} \cdot \Psi\left(x+\frac{\beta}{\eta},\, y+\frac{\gamma}{\eta},\, z+\frac{\epsilon}{\eta}\right) d\beta\,d\gamma\,d\epsilon \qquad (h)$$

If we now let $t = 0$, this becomes

$$T_0 = C(2\sqrt{\alpha})^3\Psi(x,y,z)\int_{-\infty}^{\infty} e^{-\beta^2}\,d\beta\int_{-\infty}^{\infty} e^{-\gamma^2}\,d\gamma\int_{-\infty}^{\infty} e^{-\epsilon^2}\,d\epsilon \qquad (i)$$

and, remembering that

$$\int_{-\infty}^{\infty} e^{-p^2}\,dp = \sqrt{\pi} \qquad (j)$$

this becomes
$$T_0 = C(2\sqrt{\alpha\pi})^3\Psi(x,y,z) \qquad (k)$$

From (b) we see that if

$$C = \left(\frac{1}{2\sqrt{\alpha\pi}}\right)^3 \qquad (l)$$

and
$$\Psi(x,y,z) = f(x,y,z) = f(\lambda,\mu,\nu) \quad \text{since } t = 0 \qquad (m)$$

the boundary condition (b) is fulfilled. Putting in (h) these values of C and Ψ, we find at once that it reduces to the solution (9.41g) already found.

9.43. Formulas for Various Solids. Since the solution of the heat-conduction equation for three dimensions and with constant initial temperature can in most cases be considered as the product of three solutions, each of one dimension, it is possible* to arrive at once at a solution of a large variety of simple cases where the initial and surface temperatures are each constant. Equation (8.16k) gives for the center temperature of a slab of thickness l, initially at T_0 and with surfaces at T_s,

* See Newman[102] and Olson and Schultz.[106]

the relation

$$\frac{T - T_s}{T_0 - T_s} = S\left(\frac{\alpha t}{l^2}\right) \qquad (a)$$

For the center of a rectangular brick of dimensions l, m, and n we would accordingly have

$$\frac{T - T_s}{T_0 - T_s} = S\left(\frac{\alpha t}{l^2}\right) \times S\left(\frac{\alpha t}{m^2}\right) \times S\left(\frac{\alpha t}{n^2}\right) \qquad (b)$$

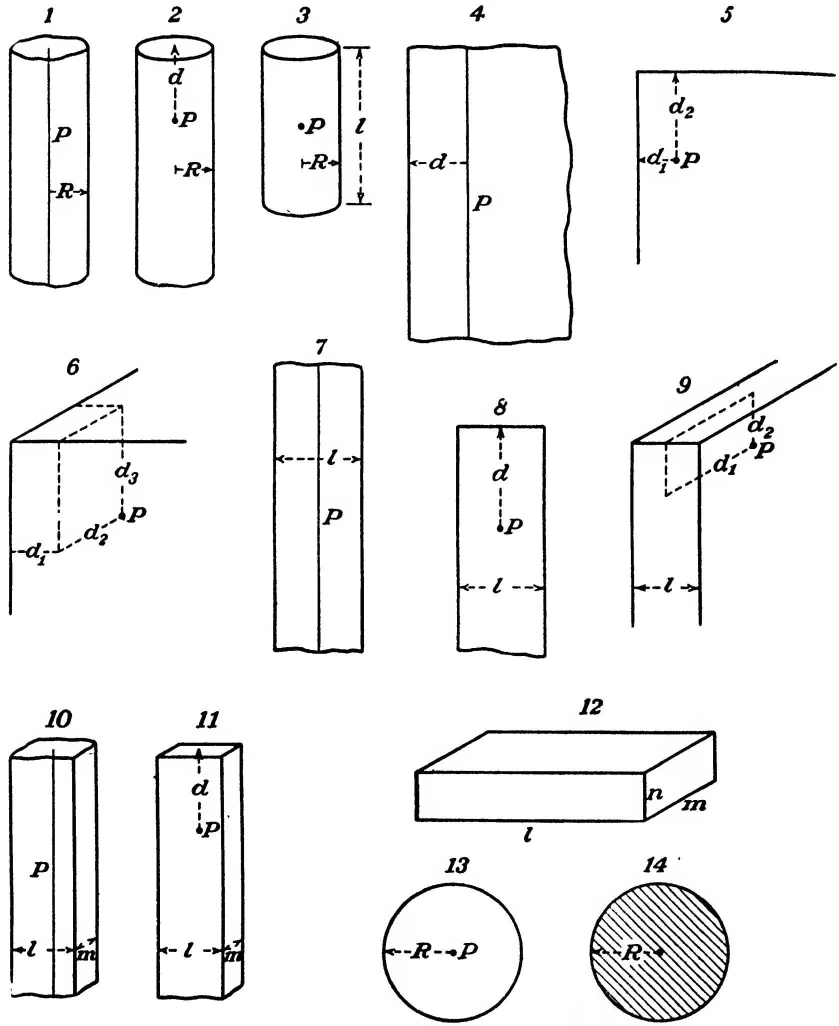

FIG. 9.3. Diagrams to accompany Table 9.1.

TABLE 9.1.—PHYSICAL EQUIVALENTS AND HEAT-CONDUCTION FORMULAS* OF A NUMBER OF SOLIDS†

Case	Physical equivalent	Point in object	$\dfrac{T - T_s}{T_0 - T_s}$
1. Infinite cylinder	Region of a long cylinder of radius R remote from both ends, or a finite cylinder with insulated ends	On axis	$C\left(\dfrac{\alpha t}{R^2}\right)$
2. Semiinfinite cylinder	Region of a long cylinder near one of the ends	On axis, distance d from end	$C\left(\dfrac{\alpha t}{R^2}\right) \times \Phi\left(\dfrac{d}{2\sqrt{\alpha t}}\right)$
3. Finite cylinder	Cylinder whose length l is of the same order of magnitude as its diameter	Geometric center	$C\left(\dfrac{\alpha t}{R^2}\right) \times S\left(\dfrac{\alpha t}{l^2}\right)$
4. Semiinfinite solid	Region near a plane face of a large solid	Distance d from surface	$\Phi\left(\dfrac{d}{2\sqrt{\alpha t}}\right)$
5. Quarter-infinite solid	Region near the edge or intersection of two perpendicular faces of a large solid	Distance d_1 from one surface, d_2 from second surface	$\Phi\left(\dfrac{d_1}{2\sqrt{\alpha t}}\right) \times \Phi\left(\dfrac{d_2}{2\sqrt{\alpha t}}\right)$
6. Eighth-infinite solid	Region near the corner or intersection of three mutually perpendicular faces of a large solid	Distance d_1 from one surface, d_2 from second surface, d_3 from third	$\Phi\left(\dfrac{d_1}{2\sqrt{\alpha t}}\right) \times \Phi\left(\dfrac{d_2}{2\sqrt{\alpha t}}\right) \times \Phi\left(\dfrac{d_3}{2\sqrt{\alpha t}}\right)$
7. Infinite slab	Region of a large slab of thickness l remote from the edges	Midway between planes	$S\left(\dfrac{\alpha t}{l^2}\right)$

8. Semiinfinite slab	Region of a large slab near one plane surface perpendicular to the faces of the slab	Midway between parallel planes, distance d from end	$S\left(\frac{\alpha t}{l^2}\right) \times \Phi\left(\frac{d}{2\sqrt{\alpha t}}\right)$
9. Quarter-infinite slab	Region of a large slab near the intersection of two perpendicular surfaces, each of which is perpendicular to the faces of the slab	Midway between parallel planes, distance d_1 from one edge, d_2 from other edge	$S\left(\frac{\alpha t}{l^2}\right) \times \Phi\left(\frac{d_1}{2\sqrt{\alpha t}}\right) \times \Phi\left(\frac{d_2}{2\sqrt{\alpha t}}\right)$
10. Infinite rectangular rod	Region of a long rod of rectangular cross section (width l and thickness m), remote from both ends, or a rectangular rod with insulated ends	On axis	$S\left(\frac{\alpha t}{l^2}\right) \times S\left(\frac{\alpha t}{m^2}\right)$
11. Semiinfinite rectangular rod	Region of a long rod of rectangular cross section, near one of the ends	On axis, distance d from end	$S\left(\frac{\alpha t}{l^2}\right) \times S\left(\frac{\alpha t}{m^2}\right) \times \Phi\left(\frac{d}{2\sqrt{\alpha t}}\right)$
12. Brick	Parallelepiped or brick whose length l, width m, and height n are of the same order of magnitude	Geometric center	$S\left(\frac{\alpha t}{l^2}\right) \times S\left(\frac{\alpha t}{m^2}\right) \times S\left(\frac{\alpha t}{n^2}\right)$
13. Sphere	Sphere of radius R	Center	$B\left(\frac{\pi^2\alpha t}{R^2}\right)$
14. Sphere	Sphere of radius R	Average temperature	$B_a\left(\frac{\pi^2\alpha t}{R^2}\right)$

* Largely from Olson and Schultz.[106]

† See Appendixes D, G, H, and J for tables of the functions Φ, S, B, and C.

while for the center of a round cylinder of radius R and length l the relation would be

$$\frac{T - T_s}{T_0 - T_s} = C\left(\frac{\alpha t}{R^2}\right) \times S\left(\frac{\alpha t}{l^2}\right) \qquad (c)$$

Table 9.1 lists the formulas for all the simpler cases.

APPLICATIONS

9.44. Canning Process. Brick Temperatures. The foregoing equations have been made use of in the canning industry in studying the time-temperature relations in the sterilizing process. In this connection we may calculate the temperature at the center of a can of vegetables of length 11.0 cm and radius 4.2 cm, after 30 min in steam at 130°C, the initial temperature being 20°C. Using the same diffusivity (0.00143 cgs) as for water, we have

$$\frac{T - 130}{20 - 130} = S(0.0213) \times C(0.146) = 0.65 \qquad (a)$$

or $T = 58.5$°C. It is to be noted in this connection that the center temperature will of course continue to rise even after the can has been removed from the boiler and the surface starts to cool.

As a second illustration we shall calculate the temperature at the center of a brick ($\alpha = 0.020$ fph) of dimensions 2 by 4 by 8 in. What is the temperature after 15 min if the brick is initially at 300°F and the surface has been chilled to 40°F? We have here

$$\frac{T - 40}{300 - 40} = S(0.18) \times S(0.045) \times S(0.011) = 0.174 \qquad (b)$$

or $T = 85$°F.

In all our previous discussions the expressions *infinite plate, long rod, point remote from end,* etc., are of frequent occurrence. It is natural to question the error involved if the dimensions do not meet these ideal specifications. The problem of the brick solved above indicates the answer. It will be noted that the heat flow in the direction of the largest dimension, which is four

times the smallest, has little effect on the result. If the largest dimension is half a dozen, or so, times the smallest, the ideal conditions may in general be considered as fulfilled.

9.45. Drying of Porous Solids. As indicated in Sec. 1.4, the diffusion of moisture in porous solids follows, within certain limits, equations similar to those for heat conduction. Newman[101] and others have developed the theory along these lines. As an example of this application, we shall solve the following problem: A sphere of clay 6 in. in diameter dries from a moisture content of 18 per cent (*i.e.*, the water is this fraction of the total weight) down to 12 per cent in 8 hr, under conditions that indicate that diffusion (*i.e.*, heat-conduction) equations apply in this case. If the equilibrium moisture content is 4 per cent, how much more time would be required for drying down to 7 per cent moisture?

In solving we must first translate the moisture-content figures to percentages of dry weight, *i.e.*, pounds of water per pound of dry clay. This gives

$$\begin{aligned}
C_0 &= \text{total initial moisture content} &&= 18/82 = 0.219\\
C_a &= \text{total moisture content at 8 hr} &&= 12/88 = 0.136\\
C_b &= \text{total final moisture content} &&= 7/93 = 0.075\\
C_s &= \text{equilibrium moisture content} &&= 4/96 = 0.042
\end{aligned}$$

In applying heat-conduction equations to diffusion problems, liquid concentration corresponds to temperature. We may accordingly use, in this case, the equations developed in Secs. 9.16 to 9.18. We must note, however, that while C_0 and C_s refer to moisture concentrations that may be assumed to be uniform throughout the sphere, this is not true for C_a and C_b, which are average* concentrations after certain drying periods. We must accordingly use the equations of Sec. 9.18. We have then

$$\frac{C_a - C_s}{C_0 - C_s}\left(\text{corresponding to } \frac{T_a - T_s}{T_0 - T_s}\right) = \frac{0.136 - 0.042}{0.219 - 0.042} = 0.531 = B_a(x) \qquad (a)$$

* A little thought will show the reason for this. Temperature is readily determined for various points in a body, but this would be difficult for liquid concentrations, which are usually measured by weighing and hence are average values.

This gives, from Appendix H,

$$x = 0.258 = \frac{\pi^2 \alpha t}{R^2} \qquad (b)$$

from which we get as the diffusion constant in this case,

$$\alpha = \frac{0.258 \times 0.0625}{8\pi^2} = 0.000204 \text{ ft}^2\text{ hr} \qquad (c)$$

For the final 7 per cent moisture content we have

$$\frac{0.075 - 0.042}{0.219 - 0.042} = 0.186 = B_a(x), \qquad \text{or } x = 1.19 \qquad (d)$$

Using the above value of α, we have for the total drying time

$$t = 36.9 \text{ hr} \qquad (e)$$

or 28.9 hr beyond the first drying period.

Tests of drying periods on one shape enable calculations of drying times for other shapes and sizes of solids made of the same material. Such calculations, however, require curves or tables (similar to our B_a table) for average temperatures or moisture contents, for such shapes as the slab, cylinder, brick, etc. For such, as well as for a more complete treatment of the subject, the reader is referred to Newman's paper.[101]

9.46. Problems

1. A square pine ($\alpha = 0.0059$ fph) post of large dimensions, at 70°F, has its surface heated to 250°F. What is the temperature 1 in. below the surface after half an hour? Solve this for a point well away from the edge and also for one near an edge and 1 in. from each surface. What bearing do these results have on the form of the isotherms near the edges? (In answering this question calculate at what equal distance from each face, near the edge, the temperature is the same as at 1 in. from the surface and well away from the edge.) *Ans.* 120°F, 156°F

2. In the brick ($\alpha = 0.0074$ cgs) of Sec. 8.26 heated for 10 min, what would the result have been if the other dimensions had been taken into account? Assume the width to be twice the thickness and the length four times. *Ans.* $0.60T_s$

3. Molten copper (use $k = 0.92$, $c = 0.091$, $\rho = 8.9$ cgs) at 1085°C is suddenly poured into a cubical cavity in a large mass of copper at 0°C. If the edge of the cube is 40 cm, find the temperature at the center after 5 min. Neglect latent heat of fusion (cf. Problem 1, Sec. 9.4). *Ans.* 186°C

4. A sphere, cylinder (height equal to diameter), and cube of cement ($\alpha = 0.04$ fph) are each of the same linear dimensions, *viz.*, 6 in. high. If the initial temperature is zero and the surface in each case is heated to 100°F, calculate the temperature in the center in each case after ½ hr. Also, make the same calculations for all bodies of the same volume, equal to that of the 6-in. cube. *Ans.* 91.4°F, 85.5°F, 80.7°F; 74.3°F, 78.5°F, 80.7°F

5. A clay ball 4 in. in diameter dries from a moisture content of 19 per cent (*i.e.*, 19 per cent of total weight) down to 11 per cent in 3 hr. Assuming that diffusion equations apply and that the equilibrium moisture content is 3 per cent, what will be the moisture content after 10 hr of drying? *Ans.* 6.3 per cent

6. Consider the steady temperature state in a long rod of radius R, one-half of whose surface for $0 < \theta < \pi$ is kept at T_s and the other half, for $\pi < \theta < 2\pi$, at zero. Since T is here a function of the cylindrical coordinates r and θ only, the Fourier equation for the steady state is*

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0 \tag{1}$$

Show that the temperature at any point (r, θ) is given by

$$T = \frac{T_s}{2}\left[1 + \frac{4}{\pi}\left\{\frac{r}{R}\sin\theta + \left(\frac{r}{R}\right)^3\frac{\sin 3\theta}{3} + \left(\frac{r}{R}\right)^5\frac{\sin 5\theta}{5} + \cdots\right\}\right] \tag{2}$$

$$= \frac{T_s}{2}\left[1 + \frac{2}{\pi}\tan^{-1}\left(\frac{\sin\theta}{\sinh \ln (R/r)}\right)\right] \tag{3}$$

$$= \frac{T_s}{\pi}\cot^{-1}\left(\frac{-\sin\theta}{\sinh \ln (R/r)}\right) \tag{4}$$

Show also that the conjugate function to T, of the complex variable $\theta + i \ln (R/r)$, which gives the lines of heat flow is

$$U = \frac{T_s}{\pi}\tanh^{-1}\left(\frac{\cos\theta}{\cosh \ln (R/r)}\right) \tag{5}$$

SUGGESTIONS. Apply the method of Sec. 4.3 and show that

$$T = \left(\frac{r}{R}\right)^n \frac{\sin n\theta}{n}$$

is a particular solution of the Fourier equation, where n may be any positive integer. Assume that the desired solution is possible with a series of such particular solutions having undetermined coefficients as in (4.2k), including a possible constant term. Choose these coefficients such that the boundary conditions at $r = R$ are satisfied, thus giving the first form of the solution above. Compare this with (4.2n) where y corresponds to $\ln (R/r)$ and get the closed forms for T. The conjugate function follows from Appendix L.

* See Churchill.[32, p. 13]

CHAPTER 10

FORMATION OF ICE

10.1. We shall now take up the study of the formation of ice, *i.e.*, of the relationship that must exist between the thickness and rate of freezing or melting of a sheet of ice and the time when a lake of still water is frozen or a sheet of ice thawed. In our previous study of the various cases of heat conduction in a medium we have assumed that the addition or subtraction of heat from any element of the medium serves only to change its temperature and does not in any way alter its conductivity constants or other physical properties. In ice formation, however, we have essentially a more complicated case, for the freezing of water or thawing of ice results not only in a change from one medium to another that has entirely different thermal constants, but also in the accompanying release or absorption of the latent heat of fusion.

10.2. We shall treat the problem in two somewhat different ways, the first following substantially the method of Franz Neumann* and the second that of J. Stefan.[138]† In each case we have initially a surface of still water lowered, as by contact with the air or some other body, to some temperature T_0, which must always be below the freezing point. There will then be formed a layer of ice whose thickness ϵ is a function of the time t. Take the upper surface of ice as the yz plane, and the positive x direction as running into the ice. Let T_1 apply to temperatures in the ice, and T_2 to the water; and similarly, let k_1, c_1, and α_1 be the thermal constants for ice, while k_2, c_2, and α_2 are those for water. It is assumed that there is no convection in the water, and the changes of volume that occur on freezing or melting are neglected.

* Weber-Riemann.[160, II, p. 117]

† See also Tamura.[143]

10.3. Neumann's Solution. Instead of one fundamental equation, as in the case of a single homogeneous medium, there will now be two, applying respectively to the ice and to the water under the ice. These are

$$\frac{\partial T_1}{\partial t} = \alpha_1 \frac{\partial^2 T_1}{\partial x^2} \text{ in the ice} \qquad (0 < x < \epsilon) \qquad (a)$$

and

$$\frac{\partial T_2}{\partial t} = \alpha_2 \frac{\partial^2 T_2}{\partial x^2} \text{ in the water} \qquad (\epsilon < x) \qquad (b)$$

The temperature of the boundary surface of ice and water (at $x = \epsilon$) must always be 0°C, and there will be continual formation of new ice. If the thickness increases by $d\epsilon$ in time dt, there will be set free for each unit of area an amount of heat

$$Q = L\rho_1 \, d\epsilon \qquad (c)$$

where L is the latent heat of fusion. This must escape upward by conduction through the ice, and in addition there will be a certain amount of heat carried away from the water below, so that the total amount of heat that flows outward through unit area of the lower surface of the ice sheet is

$$Q' = k_1 \left(\frac{\partial T_1}{\partial x}\right)_{x=\epsilon} dt \qquad (d)$$

Of this amount the quantity

$$Q'' = k_2 \left(\frac{\partial T_2}{\partial x}\right)_{x=\epsilon} dt \qquad (e)$$

flows up from the water below; hence, we obtain for our first boundary condition

$$\left(k_1 \frac{\partial T_1}{\partial x} - k_2 \frac{\partial T_2}{\partial x}\right)_{x=\epsilon} = L\rho_1 \frac{\partial \epsilon}{\partial t} \qquad (f)$$

The other boundary conditions are to be

$$T_1 = T_s = C_1 \qquad \text{at } x = 0 \qquad (g)$$
$$T_1 = T_2 = 0 \qquad \text{at } x = \epsilon \qquad (h)$$
$$T_2 = C_2 \qquad \text{at } x = \infty \qquad (i)$$

We also have three other boundary conditions derived from the fact that when $t = 0$, ϵ is fixed, while T_1 and T_2 must be given

as functions of x, the first between the limits 0 and ϵ and the last between ϵ and ∞. We shall investigate later the particular form of these functions.

10.4. The general solution of the problem for these conditions is not possible as yet, for the condition (10.3f) containing the unknown function ϵ is not linear and homogeneous, and we cannot then expect to reach a solution by the combination of particular solutions. Our method of solution then will be to seek particular integrals of (10.3a) and (10.3b) and, after modifying them to fit boundary conditions (10.3g), (10.3h), and (10.3i), find under what conditions the solution will satisfy (10.3f). This will then determine the initial values of ϵ, T_1, and T_2.

Now, as we have seen many times in the previous pages, the function $\Phi(x\eta)$ is a solution of such differential equations as (10.3a) and (10.3b). Consequently, if B_1, D_1, B_2, D_2 are constants and if $\eta_1 \equiv 1/2\sqrt{\alpha_1 t}$ and $\eta_2 \equiv 1/2\sqrt{\alpha_2 t}$,

$$T_1 = B_1 + D_1\Phi(x\eta_1) \tag{a}$$

and

$$T_2 = B_2 + D_2\Phi(x\eta_2) \tag{b}$$

are also solutions. Now, boundary condition (10.3h) means that $\Phi(\epsilon\eta_1)$ and $\Phi(\epsilon\eta_2)$ must each be constant, which will be true if $\epsilon = 0$, $\epsilon = \infty$, or if ϵ is proportional to $\sqrt{t}$. The first two of these assumptions are evidently inconsistent with (10.3h); thus, there remains only the last, which may be put in the form

$$\epsilon = b\sqrt{t} \tag{c}$$

where b is a constant we shall determine later, together with B_1, D_1, B_2, and D_2.

From the properties of $\Phi(x)$ we know that $\Phi(0) = 0$ and $\Phi(\infty) = 1$. Then fitting boundary conditions (10.3g), (10.3h), and (10.3i) in (a) and (b) with the use of (c), we find that

$$B_1 = C_1 \tag{d}$$

$$B_1 + D_1\Phi\left(\frac{b}{2\sqrt{\alpha_1}}\right) = 0 \tag{e}$$

$$B_2 + D_2\Phi\left(\frac{b}{2\sqrt{\alpha_2}}\right) = 0 \tag{f}$$

$$B_2 + D_2 = C_2 \tag{g}$$

while (a), (b), and (c) in connection with (10.3f) give

$$\frac{k_1 D_1}{\sqrt{\pi \alpha_1 t}} e^{-b^2/4\alpha_1} - \frac{k_2 D_2}{\sqrt{\pi \alpha_2 t}} e^{-b^2/4\alpha_2} = \frac{L\rho_1 b}{2\sqrt{t}} \tag{h}$$

Solving equations (d) to (g) for D_1 and D_2, we get

$$D_1 = \frac{-C_1}{\Phi(b/2\sqrt{\alpha_1})}; \qquad D_2 = \frac{C_2}{1 - \Phi(b/2\sqrt{\alpha_2})} \tag{i}$$

and, substituting these values in (h), we have finally

$$\frac{k_1 C_1 e^{-b^2/4\alpha_1}}{\sqrt{\alpha_1}\,\Phi(b/2\sqrt{\alpha_1})} + \frac{k_2 C_2 e^{-b^2/4\alpha_2}}{\sqrt{\alpha_2}\,[1 - \Phi(b/2\sqrt{\alpha_2})]} = -\frac{\sqrt{\pi}}{2} L\rho_1 b \tag{j}$$

10.5. This transcendental equation can be solved for b by the method employed in Sec. 9.27. Plot the curves

$$y = -\frac{\sqrt{\pi}}{2} L\rho_1 b \tag{a}$$

and

$$y = f(b) \tag{b}$$

where $f(b)$ represents the left-hand side of (10.4j). Then b is given as the abscissa of the intersection of the two curves. When b is found, the problem is solved, for from (10.4c) we can then express the exact relation between the thickness and time, and, having solved (10.4d) to (10.4g) for B_1, D_1, B_2, and D_2, we have from (10.4a) and (10.4b) the temperatures at any point in the water or ice.

10.6. We are now able to specify the initial conditions for which we have solved the problem, and which have up to this time been indeterminate. It follows from (10.4c) that when $t = 0$, $\epsilon = 0$, and from (10.4b) that T_2 is initially equal to $B_2 + D_2 = C_2$, everywhere except at the point $x = 0$, where it is indeterminate. This means that we have taken the instant $t = 0$ as that at which the ice just begins to form, the water being everywhere at the constant temperature C_2. Inasmuch, then, as there is no ice at time $t = 0$, the temperature T_1 must be indeterminate, as is shown by (10.4a).

10.7. In the case of freezing as just treated, C_1 is necessarily a negative and C_2 a positive quantity. By reversing the signs

and making C_1 positive and C_2 negative we have equations applicable to thawing. But thawing in this case means that a layer of water is formed on the ice and that the heat flows in from the upper surface of the water, which is then at temperature C_1. But this means that the ice and water have just changed places, so that in the case of thawing, C_1, k_1, α_1, and c_1 apply to the water, while C_2, k_2, α_2, and c_2 apply to the ice.

10.8. Stefan's Solution. Stefan simplified the conditions of the problem by assuming that the temperature of the water was everywhere constant and equal to zero. The fundamental equation (10.3*a*) then becomes

$$\frac{\partial T_1}{\partial t} = \alpha_1 \frac{\partial^2 T_1}{\partial x^2} \qquad \text{for } 0 < x < \epsilon \tag{a}$$

while the second is missing. Likewise, the boundary conditions (10.3*f*) to (10.3*i*) are simplified to

$$k_1 \left(\frac{\partial T_1}{\partial x}\right)_{x=\epsilon} = L\rho_1 \frac{\partial \epsilon}{\partial t} \tag{b}$$

$$T_1 = T_s = C_1 \qquad \text{at } x = 0 \tag{c}$$

$$T_1 = 0 \qquad \text{at } x = \epsilon \tag{d}$$

Since T_1 may be expressed as a function of both time and place, we may write its total differential

$$dT_1 = \frac{\partial T_1}{\partial x} dx + \frac{\partial T_1}{\partial t} dt \tag{e}$$

From (*d*) we see that this total differential must be zero at $x = \epsilon$, so that

$$\left(\frac{\partial T_1}{\partial t}\right)_{x=\epsilon} + \left(\frac{\partial T_1}{\partial x}\right)_{x=\epsilon} \frac{\partial \epsilon}{\partial t} = 0 \tag{f}$$

so that with the aid of (*b*) we have

$$\left(\frac{\partial T_1}{\partial t}\right)_{x=\epsilon} = -\frac{\alpha_1 c_1}{L} \left(\frac{\partial T_1}{\partial x}\right)^2_{x=\epsilon} \qquad \text{since } k = c\rho\alpha \tag{g}$$

As a special solution of (*a*) we shall examine the integral

$$T = B \int_{x\eta_1}^{\beta} e^{-\lambda^2}\, d\lambda \tag{h}$$

and see if the constants B and β can be so chosen that this solution is consistent with the conditions (b), (c), (d), and (f). We need not prove that (h) is a particular integral of (a), for we have used this type of integral many times as a solution of the Fourier equation in one dimension. Thus, we can proceed at once with our attempt at fitting it to these boundary conditions.

Condition (c) demands that

$$C_1 = B\int_0^\beta e^{-\lambda^2}\,d\lambda \tag{i}$$

which gives one relation between B and β. Condition (d) means that the two limits of the integral must be the same for $x = \epsilon$, so that

$$\beta = \epsilon\eta_1 = \frac{\epsilon}{2\sqrt{\alpha_1 t}} \quad \text{or} \quad \epsilon = 2\beta\sqrt{\alpha_1 t} \tag{j}$$

This gives the same law of thickness as found by Neumann's method of (10.4c), *viz.*, that the thickness increases with the square root of the time. However, we have not yet determined the constant β, and to do this we must use (g). The differential coefficients $\partial T_1/\partial t$ and $\partial T_1/\partial x$ are obtained from (h) after the method described in Sec. 7.16 and are

$$\frac{\partial T_1}{\partial t} = Be^{-x^2\eta_1^2}\frac{x\eta_1}{2t} \tag{k}$$

$$\frac{\partial T_1}{\partial x} = -Be^{-x^2\eta_1^2}\eta_1 \tag{l}$$

If we now put in these expressions $x = \epsilon = \beta/\eta_1$ and then apply (g), we have

$$Be^{-\beta^2}\frac{\beta}{2t} = -\frac{\alpha_1 c_1}{L}B^2 e^{-2\beta^2}\eta_1^2 \tag{m}$$

or, with the use of (i),

$$\beta e^{\beta^2}\int_0^\beta e^{-\lambda^2}\,d\lambda = -\frac{C_1 c_1}{2L} \tag{n}$$

and this equation enables us to determine β. The integral may be evaluated by expanding $e^{-\lambda^2}$ in the customary power series and performing the integration. When this result is multiplied by the series for βe^{β^2}, we get a series whose first two terms are

$$\beta^2\left(1 + \frac{2\beta^2}{3}\right) \tag{o}$$

To a first approximation, then, (n) gives

$$\beta^2 = -\frac{C_1 c_1}{2L} \tag{p}$$

Consequently, to the same degree of approximation, (j) means that

$$\epsilon^2 = -\frac{2C_1 c_1 \alpha_1 t}{L} \tag{q}$$

For the second approximation

$$\beta^2 \left(1 + \frac{2\beta^2}{3}\right) = -\frac{C_1 c_1}{2L} \tag{r}$$

from which β and consequently ϵ are readily determined.

Since C_1 is intrinsically negative, the right-hand member of the above equation is a positive quantity.

It should be noted that the same law of freezing holds in each case, *i.e.*, the proportionality of thickness with the square root of the time; the proportionality constant only is changed. Indeed, if we put $C_2 = 0$ in Neumann's solution (10.4j), it reduces at once to Stefan's solution (n), if $b = 2\beta\sqrt{\alpha_1}$. This makes the two expressions for the thickness, (10.4c) and (j), identical and shows that Stefan's solution may be regarded as only a special case of Neumann's.

10.9. Thickness of Ice Proportional to Time. Stefan also outlined the solution of one or two special cases that we shall find interesting.

Consider the expression

$$T_1 = \frac{B}{p}\left(e^{pt-qx} - 1\right) \tag{a}$$

where B, p, and q are constants.

It may be readily seen upon differentiation that if

$$p = \alpha_1 q^2 \tag{b}$$

(a) is a solution of the fundamental equation (10.8a). Now

$$T_1 = 0 \qquad \text{for } pt - qx = 0 \tag{c}$$

and from (10.8d)

$$T_1 = 0 \qquad \text{at } x = \epsilon \tag{d}$$

from which

$$pt - qx = 0 \qquad \text{at } x = \epsilon \tag{e}$$

or

$$\epsilon = q\alpha_1 t \tag{f}$$

This shows that the thickness of ice may increase in direct proportion to the time if T_s is not a constant, as we have heretofore taken it. Equation (a) shows that (since $T_1 = T_s$ when $x = 0$), T_s must be a function of the time, and it will be our task to investigate the form of this function.

Since (10.8g) must hold, we find on substitution of (a) and (f) that

$$B = -\frac{c_1\alpha_1 q^2 B^2}{p^2 L} = -\frac{c_1 B^2}{pL} \tag{g}$$

so that the relation between B and p is

$$p = -\frac{Bc_1}{L} \tag{h}$$

For $x = 0$ we find from (a) that

$$T_s = \frac{B}{p}(e^{pt} - 1) \tag{i}$$

$$= Bt - \frac{c_1}{L}\frac{B^2t^2}{2!} + \frac{c_1^2}{L^2}\frac{B^3t^3}{3!} - \cdots \tag{j}$$

This shows, since B is negative, that if the thickness of ice is to increase directly as the time, the surface temperature must decrease more rapidly than as a linear function of the time. For any value we wish to give B, the thickness is determinate from (f).

10.10. Simple Solution for Thin Ice. If we assume that the ice is thin enough so that the temperature gradient can be considered as uniform from the upper to the lower surface, we can derive at once a very simple solution; for the quantity of heat that flows upward per unit area through the ice in time dt will then be

$$-k_1 \frac{T_s}{\epsilon}\,dt \tag{a}$$

and this must equal the heat that is released when the ice increases in thickness by $d\epsilon$. Hence, we have

$$\frac{-k_1 T_s\,dt}{\epsilon} = L\rho_1\,d\epsilon \tag{b}$$

Integrating this and assuming that ϵ is zero when t is zero, we have

$$\epsilon^2 = \frac{-2T_s k_1 t}{L\rho_1} \tag{c}$$

which is identical with (10.8q). This shows that the approximation involved in (10.8q) amounts to the assumption of a uniform temperature gradient through the ice.

10.11. With the aid of some of his formulas Stefan calculated k for polar ice from the measured rates of ice formation at Assistance Bay, Gulf of Boothia, and other places, and found

$$k = 0.0042 \text{ cgs} \tag{a}$$

This value lies between the values attributed to Neumann (0.0057) and to Forbes (0.00223), and it is only slightly lower than that now accepted (0.0053; see Appendix A).

10.12. The fact that the conductivity of ice is considerably larger than that of water gives rise to an interesting phenomenon that has been noted by H. T. Barnes.[6] When ice is being frozen on still water, particularly when the surface is kept very cold as by liquid air, ice crystals grow out into the water and are found in the ice with their long axes all pointing normal to the plane of the surface. It is probable also that their conductivity is greater along this axis. "See International Critical Tables."[64,v,p. 231]

10.13. It may be noted in connection with the study of the formation of ice that the temperature of the surface, which, as we have seen, is the controlling factor as regards the rate of freezing, is determined by a variety of conditions; for, while in most climates and under most weather conditions this is largely dependent on the temperature of the surrounding air, in cases where the air is exceptionally clear so that an appreciable amount of radiation can take place to the outer space that is nearly at absolute zero, the surface of the ice may be considerably cooler than the air. Thus, the natives of Bengal, India, make ice by exposing water in shallow earthen dishes to the clear night sky, even when the air temperature is 16 to 20°F above the freezing point.*

* See Tamura.[143] See also Sec. 5.12 on "ice mines."

10.14. Applications. While problems involving latent heat have been handled in the preceding chapters, the solutions have either neglected this consideration or taken account of it by some more or less rough approximation method. With the aid of the deductions of the present chapter many of these problems could now be treated rigorously, in particular such as relate to the freezing or thawing of soil. The equations would be directly applicable to this case if the thermal constants for soil were used instead of those for ice or water, and if the latent heat of fusion of ice was modified by a factor depending on the percentage of moisture in the soil.*

The theory would also apply to many cases of ice formation in still water, for either natural or artificial refrigeration, while, as already noted, it has been used by Stefan in connection with polar ice.

10.15. Problems

1. Applying Stefan's formulas, find how long, if $T_s = -15°C$, it will take to freeze 5 cm of ice (*a*) to the first approximation, and (*b*) to the second approximation. Use $k = 0.0052$, $c = 0.50$, $\rho = 0.92$, $\alpha = 0.011$, $L = 80$ cgs for ice. *Ans.* 3.28 hr; 3.39 hr

2. Using only the first approximation of Stefan's formula, find how long it would take to thaw 5 cm deep in a cake of ice, supposing that the water remains on top, and that the top surface of water is at +15°C. Use $\alpha = 0.00143$ cgs for water. *Ans.* 12.95 hr

3. Using Stefan's first approximation formula, find how long it would take for the soil to freeze to a depth of 1 m if the average surface temperature is −10°C and the soil initially at 0°C, and if the soil has 10 per cent moisture. Use $c = 0.45$, $\alpha = 0.0049$ cgs for the frozen soil. *Ans.* 21 days

4. Assume that T_s varies with time, so that the rate of freezing of ice is constant, and that this rate is such that 5 cm will be frozen in the time determined in Problem 1*a*. Determine T_s for 1 hr, 4 hr, and 10 hr. *Ans.* −9.5°C; −41°C; −123°C

5. If $C_1 = -15°C$ and $C_2 = +4°C$ in Neumann's solution, how long will it take to freeze 5 cm of ice (cf. Problem 1)? *Ans.* 3.8 hr

* See also Sec. 7.10, Problem 5, and Secs. 7.19, 7.20, and 11.17.

CHAPTER 11

AUXILIARY METHODS OF TREATING HEAT-CONDUCTION PROBLEMS

11.1. In this chapter we shall consider various methods of solving particular heat-conduction problems other than by the classical calculations and experiments already described. Some of the methods are electrical in character, others graphical or computational. Some apply to the steady-state flow, others to the unsteady state. While the principal use of these methods is to provide a relatively quick answer to problems whose solution by rigorous analytical methods would be difficult, they also sometimes allow the handling of cases impossible of treatment by the Fourier analysis. The accuracy is in general limited mainly by the pains one is willing to take.

METHOD OF ISOTHERMAL SURFACES AND FLOW-LINES

11.2. This is a graphical method* of considerable use in treating steady-state heat conduction in two dimensions, involving the construction of an isotherm and flow-line diagram. As an illustration we shall apply it to the case of heat flow through a "square edge," *e.g.*, one of the 12 edges of a rectangular furnace or refrigerator. Figure 11.1 represents a section of such edge, with inner and outer surfaces at temperatures T_1 and T_2, respectively. The five lines roughly parallel to these surfaces, save where they bend around at the edge, are isotherms that divide the temperature difference $T_1 - T_2$ into six equal parts of value ΔT each. The heat-flow lines are everywhere at right angles (Sec. 1.3) to the isotherms, and there is a steady rate of flow q down any lane between these flow lines. For a wall of height y normal to the diagram we have for the flow down any lane across a small portion such as $ABCD$ of average length u and

* Awbery and Schofield.[5]

width v, $q = kyv\Delta T/u$. Then, if $u = v$, as is approximately the case for all the little quadrilaterals (for the diagram is so constructed, as explained later), the flow down any lane is $q = ky\Delta T$. This is the same for all lanes since ΔT is the same between any two adjoining isotherms. Careful measurement of the diagram

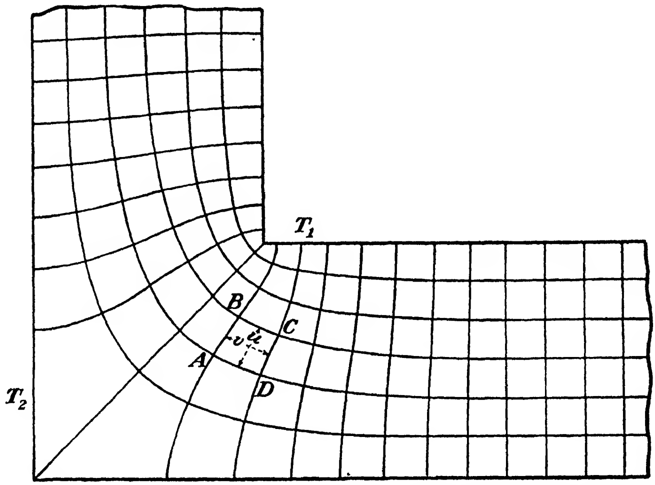

FIG. 11.1. Isotherms and flow lines for steady heat conduction through a wall near a square edge.

will show that such an edge adds approximately 3.2 lanes to the number that would be required if the spacing were uniform and equal to that remote from the edge. This means an added heat flow due to the edge of

$$3.2ky\Delta T = 3.2ky\frac{(T_1 - T_2)}{6} = 0.54kyx\frac{(T_1 - T_2)}{x}$$

where x is the wall thickness. In other words, to take account of edge loss we must add to the inside area a term $0.54yx$, where y is the (inside) length of the edge. This is in agreement with the results of Langmuir, Adams, and Meikle[81] (see Sec. 3.4).

11.3. In solving problems by this method one must first decide on the number of equal parts into which he wishes to divide the total temperature drop $T_1 - T_2$ (in this case six is used although four or five would give fairly satisfactory results) and then locate by trial the system of isotherms and flow lines *so that they intersect everywhere at right angles to form little quad-*

rilaterals that approximate squares as closely as possible; i.e., the sums of the opposite sides *should* be equal, or

$$AB + CD = BC + AD$$

When this is accomplished, the flow $ky\Delta T$ in each lane is the same between a given pair of isotherms, and, since the flow down any lane is the same throughout its length, the value of ΔT between any two adjoining pairs of isotherms must be the

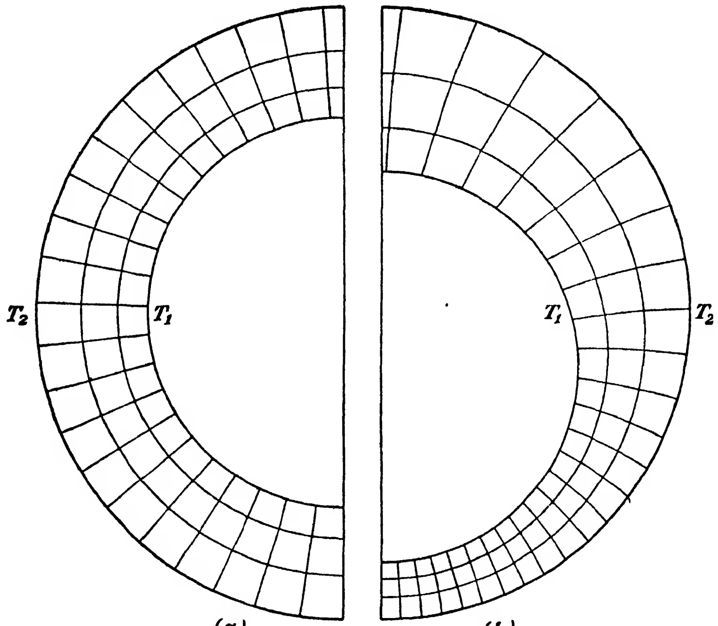

FIG. 11.2. Isotherms and flow lines for a steam pipe with (*a*) symmetrical and (*b*) nonsymmetrical coverings.

same. As explained in Sec. 11.8, a little simple electrical experimentation is useful in shortening the time required to locate the isotherms.

11.4. Nonsymmetrical Cylindrical Flow. We shall also apply this method to the problem of nonsymmetrical or eccentric cylindrical flow, *e.g.*, as in a steam pipe whose covering is thicker on one side than the other. Figure 11.2 represents two half sections of a steam pipe with a covering that in case (*a*) is symmetrical, while in (*b*) it is three times as thick on one side as

the other. Here the number of lanes in the half sections is 21.5 for the concentric case and 24.2 for the eccentric. This gives a heat loss for the eccentric case of 1.125 times that of the other, for pipe and covering proportional to the dimensions shown here, *i.e.*, radius of pipe equal to 0.64 radius of covering (cf. Sec. 11.9).

11.5. Heat Loss through a Wall with Ribs. As another illustration of this graphical method we shall apply it to the problem* of heat flow through a wall as affected by the presence of

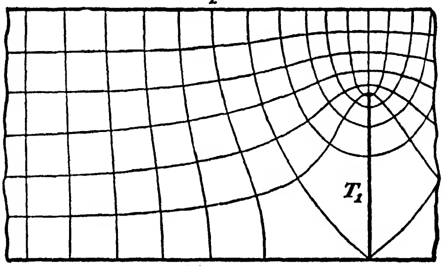

FIG. 11.3. Isotherms and flow lines for steady heat conduction through a wall with internal projecting rib of high conductivity.

internal projecting fins or ribs. It is assumed that the rib has a high conductivity as compared with the insulating material of the wall so that it is an isothermal surface taking the temperature T_1 of the surface of the wall that it joins. Figure 11.3 shows the isotherms and flow lines constructed for the case of a rib projecting two-thirds through the wall thickness. The graph shows that there are 22 lanes, *i.e.*, 11 on each side, in the region affected by the rib, while with the normal undisturbed spacing shown in the extreme left of the diagram there would be 16.6 lanes in the same length of wall. The difference or 5.4 lanes represents the heat loss due to the rib. Since each of the undisturbed lanes has a width equal to one-sixth the wall thickness, this means that such a rib, whose length is two-thirds the wall thickness, causes the same heat loss as a length of wall 5.4/6 or 0.9 the wall thickness.†

* Awbery and Schofield[5]; see also Carslaw and Jaeger.[27a, p. 359]

† For further references and methods of taking account of change of conductivity with temperature, see McAdams.[90, pp. 16, 17]

11.6. Three-dimensional Cases; Cylindrical-tank Edge Loss. The preceding cases are essentially two-dimensional in character in that the third dimension, which is perpendicular to the plane of the figure, affects the problem only as a constant factor. As a three-dimensional example we may investigate the edge losses for a heavily insulated cylindrical container with spherically shaped ends, such as is used in shipping very hot or very cold liquids, *e.g.*, liquid oxygen. Figure 11.4 represents a section of such tank covered with thick insulation. In this case the radius of the spherical end of the tank is equal to the diameter of the cylinder.

To calculate the heat loss for such a tank we shall imagine ourselves cutting a thin wedge-shaped slice, perhaps $\frac{1}{100}$ of the whole tank, by rotating the figure three degrees or so about the axis of the cylinder; we shall investigate the heat loss for this wedge. The same condition $q = kyv\Delta T/u$ holds as in the preceding cases, but here y is not constant; thus, u, instead of being equal to v, must be proportional to yv. The thickness y of the wedge is obviously proportional to the distance from the axis, and so for a constant v, as occurs in the cylinder at a point such as A well away from the ends, the distance u between isotherms is proportional to this distance from the axis. This means that the little elements, which are drawn as squares for the innermost row in the cylindrical insulation, become more and more elongated rectangles for the outer rows. A little thought will show that for the spherical ends the distance between isotherms must vary as the square of the radius of the sphere.

Figure 11.4 has been constructed to meet these various conditions as closely as possible. The proportions for the rectangles in each row have been preserved, for the cylindrical part or for the spherical part, as nearly uniform as possible when fitting around the edge. The flow down each channel that starts at the cylindrical-tank wall is the same, as in the cases previously considered, but for the spherical end the channels farthest from the axis evidently count the most because the height y obviously diminishes toward the axis. Measurement shows that the

edge loss for such an end can be taken account of by adding 33 per cent of the insulation thickness to the cylindrical length in computing the total heat loss. This means that the spherical-end loss is to be computed as the loss through the fraction of the

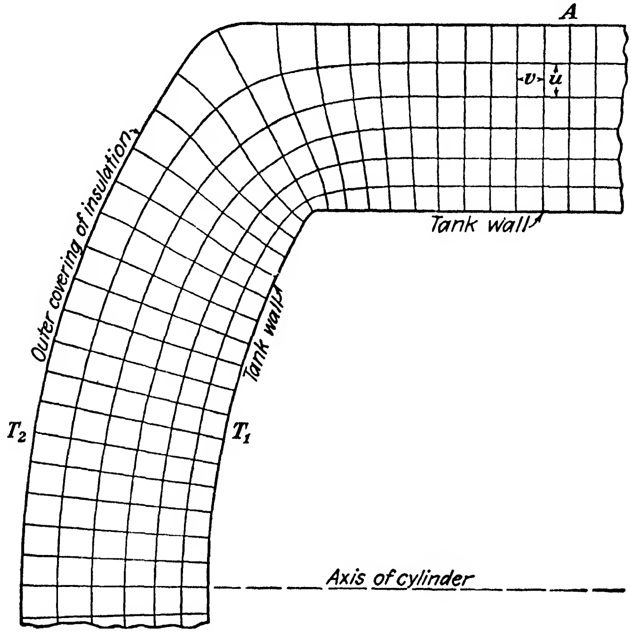

FIG. 11.4. Construction of isotherms and flow lines to show edge losses at the spherically shaped ends of a cylindrical tank (Sec. 11.6).

sphere of solid angle determined by the tank end, and the cylindrical loss computed in the usual way (Sec. 4.7), with the cylindrical length increased by two-thirds the insulation thickness to take account of the edge losses at the two ends.

ELECTRICAL METHODS

11.7. The fundamental equations for heat flow are identical with those for the flow of electricity. Ohm's law corresponds to the conduction law, potential difference to temperature difference, electrical conductivity to heat conductivity, and electrical capacity to heat capacity. This means that electrical methods can be used to solve many of the problems of heat conduction and sometimes with a great saving of time. Perhaps the most extensive application of electrical methods is in the work of

Paschkis[107,108,109] and his associates. By means of a network of resistances and condensers the electrical analogy of a heat-flow problem can be set up and a solution reached.

Much simpler electrical arrangements can be used to solve certain steady-state heat-flow problems, with k constant, such as the heat flow through the edges (cf. Secs. 3.4 and 11.2) and corners of a furnace or refrigerator. Langmuir, Adams, and Meikle[81] made measurements of the resistance of suitably shaped cells with metal and glass sides filled with copper sulphate solution, to solve these and similar problems.

A less direct method* makes use of a thin sheet of metal or layer of electrolyte in which the current is led in at one edge or several edges and out at another. The equipotential lines (corresponding to the isotherms) can then be determined and the lines of current flow (heat flow) drawn perpendicular to them.

11.8. One of the present authors has done more or less experimental work along these lines and finds that if the accuracy requirements are only moderate—*i.e.*, allowable error of a few per cent as is the case in most heat-conduction measurements—very simple arrangements will suffice. For a two-dimensional case a flat, level glass-plate cell is used with a layer of tap water 2 or 3 mm deep. Metal electrodes of the desired shape, *e.g.*, the outside and inside of a square edge (cf. Fig. 11.1), are connected with a 1,000-cycle microphone "hummer." Two metal probes or points connected with earphones are used to determine the equipotential lines. In doing this, one point is fixed and the other moved until the sound is a minimum. While the construction method described in Sec. 11.3 will, if carefully carried out, locate unambiguously the isotherms and flow lines, time may be saved by the use of the electrical method to get the form of these isotherms.

A series of measurements was also made on the resistance of cells shaped as square edges or corners, and the formulas of Langmuir (Sec. 3.4) were checked. These cells were made rather simply of metal and glass and filled with tap water with a few drops of sulphuric acid. The resistance was measured

* See *e.g.*, Schofield.[124]

with a Wheatstone-bridge circuit, the hummer being used as a battery and phones in place of galvanometer.

11.9. Eccentric Spherical and Cylindrical Flow. With the aid of simple apparatus like this a rather important problem that presents considerable analytical difficulty was solved. This is the question—already treated graphically in Sec. 11.4 for the cylindrical case—of heat flow between eccentric cylindrical or spherical surfaces. The apparatus consisted of a cell (for the cylindrical case) with glass bottom, to which was waxed a brass cylinder of 19.73 cm (7.76 in.) inside diameter. Cylinders of outside diameter 0.63, 4.92, 12.70, and 17.83 cm were used in turn as the inner electrode and the cell were filled to a depth of 16 cm with tap water. In the case of the sphere the outer shell was of 25.5 cm inside diameter, and the inner spheres of 3.81, 11.41, and 15.41 cm outside diameter, respectively. In each case the internal cylinder or sphere could be moved from the concentric position to any other within the limits. Capacity effects gave little trouble except in the cases of the larger internal cylinders or spheres. Resistances were measured with a Wheatstone-bridge circuit as mentioned above.

TABLE 11.1.—RELATIVE HEAT LOSSES FOR ECCENTRIC CYLINDERS AND SPHERES*

Insulation thickness on thin side, per cent	Cylinders				Spheres	
	$r = 0.03R$	$r = 0.25R$	$r = 0.64R$	$r = 0.90R$	$r = 0.15R$	$r = 0.60R$
100	1.00	1.00	1.00	1.00	1.00	1.00
90	1.00	1.00	1.01	1.01	1.00	1.01
80	1.01	1.01	1.02	1.02	1.00	1.02
70	1.02	1.03	1.04	1.04	1.01	1.03
60	1.05	1.05	1.08	1.08	1.02	1.05
50	1.08	1.10	1.13	1.14	1.03	1.08

* Based on resistance measurements.

The results are summarized in Table 11.1, which shows that if the internal cylinder or sphere is shifted from the concentric

position (100 per cent) until the insulation thickness on the thin side is reduced to 50 per cent of its initial value (*i.e.*, is three times as thick on one side as on the other), the heat loss will be increased by some 3 to 14 per cent according to the relative sizes of the internal cylinder or sphere (radius r) and the external one (radius R). It shows, furthermore, that the effect is less when the internal cylinder or sphere is small relative to the external one, and that it is less for the sphere than for the cylinder. The measured 13 per cent increase in the lowest line of column 4 ($r = 0.64R$) is to be compared with the 12.5 per cent obtained by the graphical solution of the problem in Sec. 11.4.

It may be pointed out that these results may be applied at once to problems involving electrical capacity, *e.g.*, a coaxial cable with eccentric core.

SOLUTIONS FROM TABLES AND CURVES

11.10. A number of tables for determining temperatures in the unsteady (*i.e.*, transient or building-up) state of heat flow are available, and one of the most useful, taken from Williamson and Adams,[161] is reproduced in Table 11.2, which is, in effect, a brief synopsis of Table 9.1. This allows the determination of the temperature T at the center of solids of various shapes, initially at temperature T_0 uniform throughout the solid, t sec (cgs) or hr (fph) after the surface temperature has been changed to T_s.

From Table 11.2 we can conclude that if a sphere of granite ($\alpha = 0.016$ cgs) of radius 15 cm and at a temperature of $T_0 = 100°C$ has its surface temperature suddenly lowered to $T_s = 0°C$, the center temperature 4,500 sec later

$$\left(i.e., \frac{\alpha t}{b^2} = 0.32\right)$$

will be $T = 8.5°C$. If $T_0 = 0°$ and $T_s = 100°C$, the temperature after 4,500 sec will be 91.5°C.

11.11. Charts. A large number of charts,* of which the best known are the Gurney-Lurie,[54] are available for the ready calcu-

* See, *e.g.*, McAdams,[90, pp. 32 ff] Ede.[35a]

lation of temperatures in slabs, cylinders, spheres, etc. These apply not only to the case of constant surface temperature but also for known temperature of surroundings with various surface coefficients of heat transfer.

TABLE 11.2.—VALUES OF $(T - T_s)/(T_0 - T_s)$ AT THE CENTER OF SOLIDS OF VARIOUS SHAPES

$\alpha t/b^2$*	Slab	Square bar	Cube	Cylinder of infinite length	Cylinder of length = diam.	Sphere
0	1	1	1	1	1	1
0.032	0.9998	0.9997	0.9995	0.9990	0.9988	0.9975
0.080	0.9752	0.9510	0.9274	0.9175	0.8947	0.8276
0.100	0.9493	0.9012	0.8555	0.8484	0.8054	0.7071
0.160	0.8458	0.7154	0.6051	0.6268	0.5301	0.4087
0.240	0.7022	0.4931	0.3462	0.3991	0.2802	0.1871
0.320	0.5779	0.3340	0.1930	0.2515	0.1453	0.0850
0.800	0.1768	0.0313	0.00553	0.0157	0.00277	0.00074
1.600	0.0246	0.00060	......	0.00015		
3.200	0.00047					

* b is the radius or half thickness.

As was made clear in Table 9.1, there are a number of cases in which the results for two- or three-dimensional heat flow may be obtained directly from the one-dimensional case. Thus, for the case of the brick-shaped solid, as shown in Sec. 9.43* the solution is readily obtained by multiplying together the three solutions for slabs whose thicknesses are the three dimensions of the brick. It is to be noted in Table 11.2 that the values for the square bar are the squares of the slab values, while those for the cube are the cubes. Also, the short-cylinder values are the product of those for the long cylinder and the slab.

THE SCHMIDT METHOD

11.12. It is possible to arrive at an approximate solution of an unsteady-state heat-conduction problem by methods, graphical or otherwise, involving only the simplest mathematics. The accuracy depends on the number of steps used in the solu-

* See also Newman.[102]

tion. Many a problem whose exact analytical solution is very difficult can be solved in this way with an accuracy sufficient for all practical purposes.

The best known approximation method is the graphical Schmidt method.[123]* As an illustration of this we shall con-

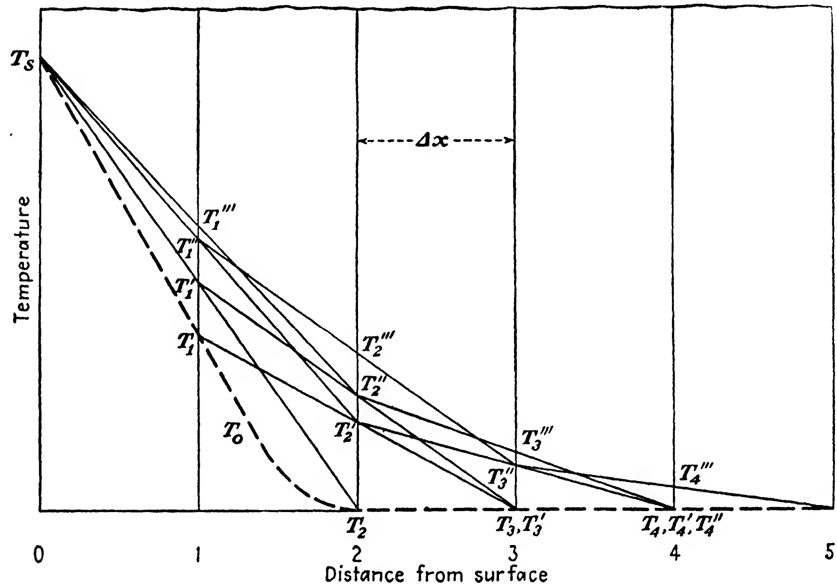

FIG. 11.5. Application of the Schmidt graphical method to one-dimensional unsteady-state heat flow in a semiinfinite solid whose initial temperature is given by the dashed line, with surface at temperature T_s.

sider one-dimensional nonsteady heat flow in a body whose plane face is at temperature T_s (*i.e.*, case of semiinfinite solid). Imagine a series of planes Δx apart in the body and let the initial temperature T_0 be represented by the heavy dashed line in Fig. 11.5. As a matter of fact, the temperature distribution might be anything, *e.g.*, $T_0 = 0$, but, for reasons that will appear in connection with the next illustration, it is somewhat easier to explain the process with a distribution of the type given here.

The average initial temperature gradient in the first layer is $(T_s - T_1)/\Delta x$, and in the second, $(T_1 - T_2)/\Delta x$. Then, in

* See also Sherwood and Reed,[129, p. 241] Fishenden and Saunders,[39, p. 77] McAdams,[90, p. 39] and Nessi and Nissole.[100] For a precursor of this method see Binder.[14]

time Δt the heat flow per unit area from the surface to plane 1 will be $k\Delta t(T_s - T_1)/\Delta x$ heat units, while $k\Delta t(T_1 - T_2)/\Delta x$ heat units will flow away from plane 1 to plane 2. The difference will remain in the vicinity of plane 1 and will heat a layer Δx thick that centers on plane 1. Then,

$$\frac{k\Delta t(T_s - T_1)}{\Delta x} - \frac{k\Delta t(T_1 - T_2)}{\Delta x} = c\rho\Delta x(T_1' - T_1) \qquad (a)$$

where T_1' is the temperature in plane 1 at time Δt (T_1'' is likewise the temperature in the same plane at time $2\Delta t$, T_4''' the temperature in plane 4 at time $3\Delta t$, etc.). This gives

$$\frac{T_s + T_2}{2} - T_1 = (T_1' - T_1)\frac{(\Delta x)^2}{2\alpha\Delta t} \qquad (b)$$

Now if Δt is taken of such size that

$$\frac{(\Delta x)^2}{2\alpha\Delta t} = 1 \qquad i.e., \ \Delta t = \frac{(\Delta x)^2}{2\alpha} \qquad (c)$$

we have

$$T_1' = \frac{T_s + T_2}{2} \qquad (d)$$

This means that the temperature in plane 1 at time Δt is the arithmetic mean of the temperatures in planes 0 and 2 at time 0. In the same way it can be shown that *the temperature in any plane at any time is the arithmetic mean of the temperatures in the planes on each side of it that prevailed Δt previously.*

This choice of Δt as determined by (c) is the principle of the Schmidt method. Figure 11.5 illustrates how the lines are drawn to determine the arithmetic means and therefore give the temperatures in the different planes for various intervals, in this case for times up to $3\Delta t$. Particular care must be taken in constructing such a diagram to see that the lines connect only points representing the same time interval, *e.g.*, T_1'' and T_3'', etc. The temperature at time $3\Delta t$ would be represented approximately by drawing a smooth curve through the points T'''.

11.13. Cooling Plate. The Schmidt method lends itself particularly well to calculations on the slab or plate. As an illustration the graph is worked out in Fig. 11.6 for a plate initially at a uniform temperature T_0 whose surfaces are sud-

denly lowered to T_s. The plate is considered as divided into 10 layers, but, because of symmetry, only half of it need be represented. Obviously, the temperatures could be reversed so that the problem is one of heating instead of cooling.

Two points are to be noted here that did not appear in connection with the graph of Fig. 11.5. The first is that here the

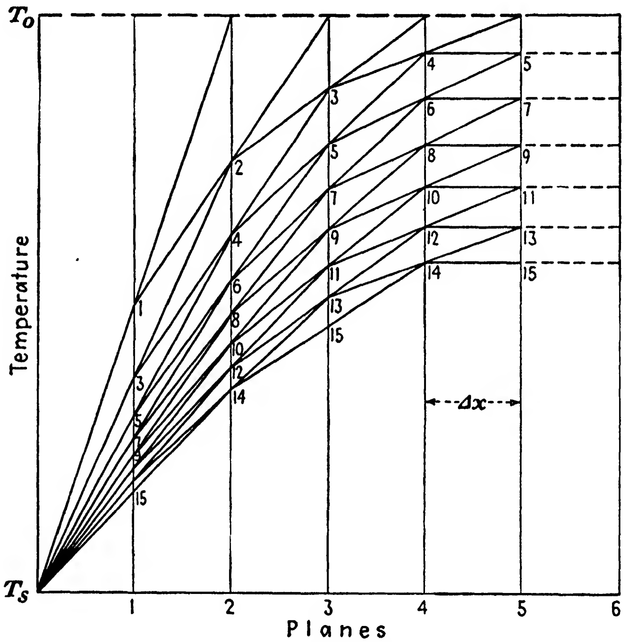

FIG. 11.6. The Schmidt graphical method applied to the cooling of a plate initially at temperature T_0. The center of the plate is at plane 5.

temperatures change only every other period. A little experience with these graphs will show that this is inherent in the construction when the initial temperature is uniform throughout the solid. This is a matter of little moment since a smooth curve, using a little interpolation, can always be drawn. The second matter is in connection with the determination of the center temperatures, plane 5 in this case. Because of symmetry the temperatures in plane 6 are identical with those in plane 4. Accordingly, the points in 5 are determined by connecting corresponding points in 4 and 6; *e.g.*, point 9 in plane 5 is found by connecting the two points 8 in planes 4 and 6.

It is of interest to compare the conclusions from Fig. 11.6 with the results of classical theory. Let us use as an example a large steel plate 1 ft thick at a temperature of 1000°F with surfaces suddenly lowered to 0°F; assume average diffusivity for this temperature range, 0.40 fph. Since each of the 10 layers is 0.1 ft in thickness, the time interval from (11.12*c*) is $\Delta t = 0.01/0.80 = 0.0125$ hr. The time t at the end of the fifteenth interval is then 0.1875 hr. From (8.16*n*) we can at once calculate the temperature of the center of the plate for this time as 607°F, while Fig. 11.6 gives about 575°F. Obviously, division into thinner layers will give more accurate results.

The Schmidt method is also capable of handling many variations of the simple-slab case,* and Nessi and Nissole[100] have worked out methods by which it is possible to apply it to cylindrical and spherical bodies. Its field of greatest usefulness, however, is the case of linear flow with thermal constants not dependent on temperature. When applicable it is probably the simplest approximation method.

THE RELAXATION METHOD

11.14. This method is an ingenious application by Emmons[36,37] of the relaxation method of Southwell.[31,137] It is applicable to one-, two-, or three-dimensional problems for either the steady or unsteady state of heat conduction and, for one dimension, is practically identical with the Schmidt method. It is particularly useful in giving quick and reasonably accurate solutions of problems involving shapes such as edges, etc., not easily treated by other methods.

We shall illustrate the use of this method by a single simple example† of steady two-dimensional flow. This is the loss from a square edge already treated in Secs. 3.4, 11.2, and 11.8. Figure 11.7 represents a section near the square edge of a rectangular furnace 24 by 24 in. inside, with a wall 10 in. thick. The inside surface of the wall is at a temperature of 500°F and the outside at 100°F. It is desired to find the temperature at a

* See McAdams,[90, p. 42] Sherwood and Reed.[129, p. 250]

† Emmons.[37, p. 609]

series of mid-points A, B, C, and D in the wall, and the heat loss from the furnace.

We shall assume that the heat is effectively conducted not by the continuous material of the wall but along a series of "rods" from point to point, forming a square lattice as indicated. When a steady state of heat transfer is reached, there will be a balance between the heat flowing to and away from one of the points A, B, etc., and we shall endeavor to fix the temperatures

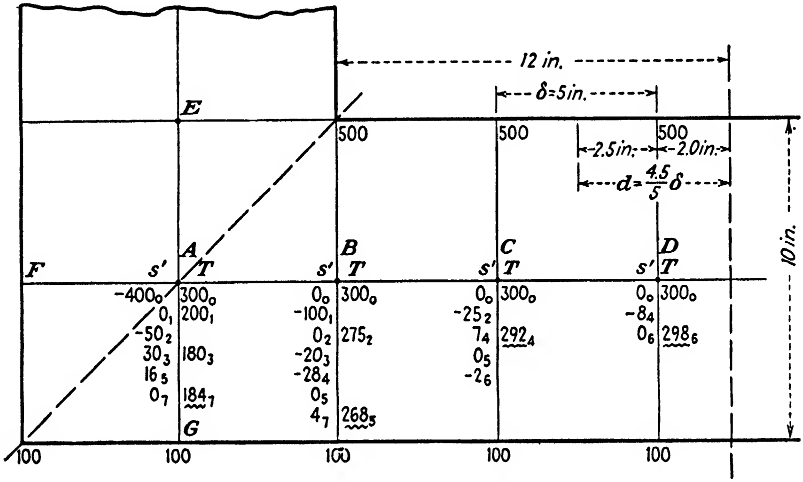

FIG. 11.7. The Emmons relaxation method applied to calculate the steady heat flow through a square edge of a rectangular furnace.

of these points so that this will be the case. Until this is done, however, there will be temperature arrangements in which more heat is conducted to a point than is taken away, in which case a positive *heat sink* of magnitude s'* will be required, while the reverse means a negative sink. Since each of these points in the plane is connected with four others, a lowering of its temperature by 1° means heat coming in from the surrounding four points with a gradient of 1° in distance δ, requiring a heat sink of magnitude 4; *i.e.*, the numerical change in the heat sink at a point is four times the temperature change of the point.

* The unit here is the amount of heat that would flow along a rod in unit time with unit temperature difference between its ends.

11.15. In explaining Fig. 11.7 we must first give each of the points A, B, C, and D a mid-temperature of 300°F. The subscript 0 indicates that this is the initial step, and the subscripts 1, 2, 3, . . . show subsequent successive steps. It is evident that this initial assumption means a balance between inflow and outflow for B, C, and D, as indicated by $s' = 0$; but while A is receiving nothing from B or E, it is losing heat to F and G under a 200°F temperature drop, and this means a (negative) sink of magnitude -400. Accordingly, we "relax" the temperature of A by subtracting 100°F, which reestablishes the heat balance so that s' is now 0. But this destroys the balance for B that now must have a sink of magnitude -100, so that the second step is to relax B by lowering its temperature 25°F. This, since it applies equally to E, requires a sink of -50 for A, likewise a sink of -25 for C, but it reduces the sink at B to 0. The third step is to lower A 20°F more, which results in a positive sink of 30 at A but a negative sink of -20 at B. The fourth step is a lowering of 8°F for C, which raises its sink to $+7$ but gives a -8 sink to D and lowers the sink at B to -28. The remaining steps are clearly indicated, and after the heat sinks are reduced to 0 or negligibly small values, the temperatures arrived at are those underlined.

To calculate the heat flow for a section of furnace 1 ft high we note that each rod, save the one through D, effectively carries the heat from an area 1 ft high and 5 in. wide. The heat transferred in unit time along the rod running from the inside surface of the wall through B would then be

$$Q = k\,\frac{5 \times 12}{144} \times \frac{(500 - 268)}{5/12} = 232k \text{ heat units} \qquad (a)$$

Thus, the total transfer through one side, including edge, would be

$$Q = 2k\left(232 + 208 + \frac{4.5}{5}\,202\right) = 1{,}244k \text{ units} \qquad (b)$$

(Note that there is no transfer considered through A since no rods from the inside pass through A.) The loss through a slab 2 ft long, 1 ft high, and 10 in. thick, with a temperature differ-

ence of 400°F would be

$$Q = k\frac{2 \times 400}{10/12} = 960k \text{ units} \tag{c}$$

The edge then increases the loss in the ratio 1,244/960 = 1.296. The Langmuir formula (Sec. 3.4) gives a ratio in this case of 1.224, which is in satisfactory agreement considering the few points used. With a finer net, *i.e.*, more points, a greater accuracy is naturally attained. For further illustrations of this interesting and useful method the reader is referred to the Emmons papers.[36,37]

THE STEP METHOD

11.16. There are a number of other approximation methods for the solution of heat-conduction or similar problems, all more or less related to the preceding but in general more complicated. We shall complete our discussion by describing in some detail a simple scheme of wide applicability for handling specific numerical problems, which will be referred to as the "step method." This consists in imagining the body divided into layers and the time into discrete intervals. The temperature throughout any layer is considered uniform and constant throughout any interval and the heat flow from layer to layer is computed, and from this the corresponding temperature change. There is nothing original in the principle of this method; like the replacement of an integral by a series it is a procedure that almost everyone has had to make use of at one time or another. It involves the same principles as the Schmidt method but lacks its ingenuity. On the other hand, its field of application is wider. It will handle problems involving changes in thermal constants with temperature, release of latent heat of fusion as in ice formation, etc., which would be difficult of solution in any other way.

While the step method is exceedingly simple in principle, there are a number of factors that must be taken into account in its application if one wants to secure best results. Accordingly, we shall illustrate it by using it in solving a variety of problems.

* Carlson,[25] Dusinberre,[35] Frocht and Leven,[44] Shortley and Weller,[130] and Thom.[146]

APPLICATIONS OF STEP METHOD

11.17. Ice Formation about Pipes; Ice Cofferdam. Our first and simplest illustration will be a problem in ice formation.* Certain open-pit mining and dam-construction operations[43,48] in wet soil have been carried on by first driving a circle of pipes into the soil and then, by introducing cold brine or other coolant, freezing a cylinder of ice about each pipe until they unite to form a circular cofferdam. We shall calculate the time required to freeze cylinders of various sizes. The same principles will of course apply to almost any case of ice formation about pipes.

Let us assume a long 4-in. pipe (outside radius 5.72 cm) driven into soil of temperature 0°C. Assume a 50 per cent (by volume) water saturation and a latent heat of fusion of 40 cal/cm³. The outside of the pipe is kept at $-T_0$°C, and, since specific-heat considerations are secondary here to latent heat, it is assumed† that the temperature distribution and heat flow are similar to those in the steady state. As the ice is formed, the latent heat released is conducted radially through the frozen-soil cylinder (assumed conductivity 0.0045 cgs) to the central pipe.

Call r_2 the radius of the frozen-soil cylinder at the beginning of any time interval Δt and r_3 the radius at the end. The average radius $r_a = (r_2 + r_3)/2$, the volume of the cylindrical layer of frozen soil, per cm cylinder length, is $\pi(r_3^2 - r_2^2)$, and the latent heat released is $40\pi(r_3^2 - r_2^2)$ cal/cm. Applying (4.6*f*) for the steady state of radial conduction per cm length of a cylinder, we have for the heat transfer in Δt sec,

$$Q = 2\pi \frac{0.0045 T_0 \Delta t}{2.303 \log_{10} r_a/5.72} = 40\pi(r_3^2 - r_2^2) \text{ cal/cm} \qquad (a)$$

This gives, if Δt is in days,

$$T_0 \Delta t = \frac{20(r_3^2 - r_2^2) \times 2.303 \log_{10} r_a/5.72}{0.0045 \times 86{,}400}$$

$$= 0.1185(r_3^2 - r_2^2) \log_{10} \frac{r_a}{5.72} \qquad (b)$$

* For the analytical solution of this problem see Pekeris and Slichter.[110]

† Pekeris and Slichter.[110, p. 135]

TABLE 11.3.—STEP CALCULATIONS FOR THE FREEZING OF WET SOIL AT 0°C ABOUT A PIPE OF EXTERNAL RADIUS 5.72 CM AT TEMPERATURE $-T$°C

A	B	C	D	E	F	G	H	I	J	K	L	M
Layer	r_2, cm	r_3, cm	r_a, cm	$r_3^2 - r_2^2$	$\log_{10} \frac{r_a}{5.72}$	$T_0 \Delta t =$ 0.1185EF	Δt days for T_0 = 15°C, days	t days for T_0 = 15°C, days	$T_0 = 24 \sin \left(\frac{\pi N}{195}\right)$, °C	Δt days $\left(= \frac{G}{J}\right)$, days	t, days	t,* days
1	5.72	10.72	8.22	82.2	0.157	1.530	0.102	0.102	For $N = 1$, $T_0 = 0.387°$ $N = 2$, $T_0 = 0.773$ $N = 1.4$, $T_0 = 0.539$	3.95 1.98 2.84	2.8	
2	10.72	15.72	13.22	132.2	0.364	5.71	0.381	0.483	$N = 5$, $T_0 = 1.93$ $N = 4$, $T_0 = 1.55$ $N = 4.5$, $T_0 = 1.74$	2.96 3.69 3.28	6.1	
3	15.72	25.72	20.72	414	0.559	27.4	1.825	2.308	$N = 9$, $T_0 = 3.47$ $N = 10$, $T_0 = 3.85$ $N = 9.8$, $T_0 = 3.78$	7.89 7.12 7.25	13.4	11
4	25.72	35.72	30.72	614	0.730	53.2	3.55	5.86	$N = 18$, $T_0 = 6.82$ $N = 17$, $T_0 = 6.50$ $N = 17.4$, $T_0 = 6.65$	7.80 8.18 8.00	21.4	20
5	35.72	55.72	45.72	1,829	0.902	195.5	13.02	18.88	$N = 29$, $T_0 = 10.8$ $N = 30$, $T_0 = 11.2$	18.0 17.5	38.9	37

TABLE 11.3.—(*Continued*)

A	B	C	D	E	F	G	H	I	J	K	L	M
Layer	r_2, cm	r_3, cm	r_a, cm	$r_3 - r_2^2$	$\log_{10} \frac{r_a}{5.72}$	$T_0\Delta t = 0.1185EF$	Δt days for $T_0 = 15°C$, days	t days for $T_0 = 15°C$, days	$T_0 = 24 \sin \left(\frac{\pi N}{195}\right)$, °C	Δt days $\left(= \frac{G}{J}\right)$, days	t, days	t* days
6	55.72	75.72	65.72	2,630	1.060	331	22.08	40.96	$N = 50$, $T_0 = 17.3$ $N = 49$, $T_0 = 17.0$	19.1 19.5	58.4	58
7	75.72	105.72	90.72	5,450	1.200	775	51.7	92.7	$N = 72$, $T_0 = 22.0$ $N = 75$, $T_0 = 22.4$ $N = 76$, $T_0 = 22.6$	35.2 34.6 34.3	92.7	91
8	105.72	135.72	120.72	7,240	1.324	1,136	75.8	168.5	$N = 120$, $T_0 = 22.4$ $N = 118$, $T_0 = 22.7$	50.7 50.1	142.8	145
9	135.72	175.72	155.72	12,460	1.435	2,120	141.4	309.9				
10	175.72	215.72	195.72	156,500	1.534	2,845	189.7	499.6				

* Pekeris and Slichter.[110]

Two calculations will be made: one for T_0 constant and the other for $T_0 = 24 \sin (\pi N/195)°C$* corresponding to a case where the cooling of the liquid, which is circulated through the pipes, is provided naturally by winter temperatures. N is the number of elapsed days since the beginning of the freezing process, and the angle in brackets is measured in radians.

The step calculations are given in Table 11.3. The first layers of ice are taken as 5 cm thick, then 10, 20, 30, 40 cm, respectively. It will be noted in columns J and K that a certain amount of trial and error is involved in arriving at the value of N and the corresponding Δt, for T_0 must obviously be taken as the mean temperature for the period under consideration. This means that N must come out as the approximate average of the initial and final times for the period, or, in other words, for any layer the acceptable value of N must approximately equal one-half the Δt (column K) for that layer plus the t (column L) of the preceding layer. The last listed is the accepted value. The values in column M are calculated by Pekeris and Slichter.

The results are seen to be in satisfactory agreement with the Pekeris and Slichter calculations. It is to be noted from (*a*) and (*b*) that, for the case of a constant T_0, the time required for the freezing of any particular size of cylinder is directly proportional to the latent heat of fusion, *i.e.*, to the moisture content of the soil. It is also inversely proportional to the conductivity of the frozen soil and inversely proportional to T_0. By the use of extreme cooling measures the time required for the production of the largest cylinder here considered might be reduced to a very few months. This assumes a conductivity independent of temperature, which in general would not be the case. A more exact solution, taking account of such variation in conductivity and also of specific-heat considerations, could be obtained as well by the step method.

11.18. Semiinfinite Solid; Warming of Soil. The step method will now be applied to the problem of one-dimensional heat flow from a warm surface into a solid at a cooler uniform temperature. While one would not in general apply the step

* Pekeris and Slichter.[110, p. 137]

method to a problem for which the solution is so readily available by analytical means—or for that matter by the simple Schmidt method—it nevertheless serves in this case as a good illustration.

We shall determine the temperatures at various depths and times in soil (assume $k = 0.0037$; $c = 0.45$; $\rho = 1.67$; $\alpha = 0.0049$ cgs) initially at 0°C, whose surface is suddenly warmed to 10°C.

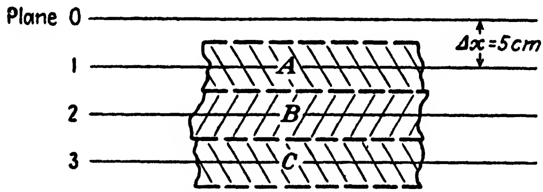

FIG. 11.8. Application of the step method to the problem of warming a semi-infinite solid.

Imagine horizontal planes in the soil 5 cm apart (Fig. 11.8) and let us inquire what will happen in the first 1,000 sec. In this period it is assumed that heat flows from a surface at 10°C through a layer of soil 5 cm thick to plane 1 at 0°C. This amount of heat per square centimeter of area is

$$Q = 1{,}000 \times 0.0037 \times {}^{10}\!/_{5} = 7.4 \text{ cal}$$

It will go toward warming up a layer (A in Fig. 11.8) 5 cm thick, centered on plane 1, and its temperature will rise by

$$\frac{Q}{5c\rho} = 1.97°\text{C}$$

In the second interval, which is taken as 1,500 sec, heat will flow from plane 0 to plane 1 under an initial temperature difference of 8.03°C and from plane 1 to plane 2 under a difference of 1.97°C. The heat delivered to plane 1 in this interval is 8.91 cal, of which 2.18 cal flows on to plane 2, leaving 6.73 cal to increase the temperature of plane 1 (A) by 1.79°C, while plane 2 (B) rises to 0.58°C. This is the temperature in plane 2 at the end of 2,500 sec, while plane 1 is at $1.97 + 1.79 = 3.76$°C.

The step calculations are given in Table 11.4. Here Δt is the magnitude of the interval and t the total elapsed time at the end of the interval. ΔT is the temperature difference between

TABLE 11.4.—STEP CALCULATIONS FOR LINEAR HEAT FLOW INTO SOIL AT 0°C WITH SURFACE AT 10°C. $\Delta x = 5$ CM, $k = 0.0037$, $c\rho = 0.752$ CGS

A	B	C	D	E	F	G	H	I
Plane	Δt, sec	t, sec	ΔT, °C	Q $(= k\Delta t\Delta T/\Delta x)$, cal/cm²	$Q_m - Q_n$, cal/cm²	δT $\left(= \frac{Q_m - Q_n}{c\rho\Delta x}\right)$, °C	T, °C	T_f (formula), °C
1	1,000	1,000	10	7.40	7.40	1.97	1.97	1.11
1	1,500	2,500	8.03	8.91	6.73	1.79	3.76	3.13
2	''	''	1.97	2.18	2.18	0.58	0.58	0.43
1	2,000	4,500	6.24	9.23	4.47	1.19	4.95	4.51
2	''	''	3.18	4.70	3.86	1.03	1.61	1.31
3	''	''	0 58	0.84	0.84	0.22	0.22	0 24
1	2,500	7,000	5.05	9 35	3.17	0.84	5.79	5.45
2	''	''	3 34	6.18	3.61	0.96	2 57	2.28
3	''	''	1 39	2.57	2.04	0.54	0.76	0.70
4	''	''	0 22	0.41	0.41	0.11	0.11	0.16
1	2,500	9,500	4.21	7 80	1.84	0.49	6.28	6.05
2	''	''	3.22	5.96	2.61	0.69	3 26	3.02
3	''	''	1.81	3.35	2.15	0.57	1.33	1.21
4	''	''	0 65	1 20	1.00	0.27	0.38	0.38
5	''	''	0.11	0.20	0.20	0.05	0.05	0.10
1	4,000	13,500	3.72	11 03	2.09	0.56	6.84	6.64
2	''	''	3.02	8.94	3 22	0.85	4 11	3.86
3	''	''	1.93	5.72	2.91	0.77	2.10	1.92
4	''	''	0.95	2.81	1 83	0.49	0 87	0.82
5	''	''	0.33	0.98	0.83	0.22	0 27	0.29
6	''	''	0 05	0.15	0.15	0 04	0 04	0.09
1	5,000	18,500	3.16	11.70	1.60	0.42	7.26	7.11
2	''	''	2.73	10.10	2.70	0.72	4 83	4.57
3	''	''	2.00	7.40	2.85	0.76	2.86	2.65
4	''	''	1.23	4.55	2 33	0.62	1 49	1.37
5	''	''	0.60	2.22	1.37	0.36	0.63	0.63
6	''	''	0.23	0.85	0.70	0.19	0.23	0.26
7	''	''	0.04	0.15	0 15	0.04	0.04	0.09
1	7,500	26,000	2 74	15.20	1.70	0.45	7.71	7.54
2	''	''	2.43	13.50	2 56	0.68	5.51*	5.31
3	''	''	1.97	10.94	3 33	0.89	3.75	3 48
4	''	''	1.37	7.61	2.83	0.75	2.24*	2.10
5	''	''	0 86	4.78	2.56	0.68	1.31	1.17
6	''	''	0.40	2.22	1.17	0.31	0.54*	0.60
7	''	''	0.19	1.05	0 83	0.22	0.26	0.29
8	''	''	0.04	0 22	0.22	0 06	0.06*	0.12
				Change to Δx = 10 cm				
2	12,000	38,000	4 49	19 93	5 39	0 72	6.23	6.05
4	''	''	3.27	14.54	6.99	0.93	3.17	3.00
6	''	''	1 70	7 55	5.42	0.72	1.26	1.21
8	''	''	0 48	2 13	1.86	0 25	0.31	0.38
10	''	''	0.06	0 27	0 27	0 04	0.04	0.09
2	20,000	58,000	3.77	27.90	5 27	0 70	6 93	6 86
4	''	''	3.06	22.63	8 49	1 13	4.30	4.03
6	''	''	1.91	14.14	7.11	0.95	2.21	2.08
8	''	''	0.95	7.03	5.03	0.67	0.98	0.94
10	''	''	0 27	2.00	1 70	0 23	0.27	0.36
12	''	''	0.04	0.30	0.30	0.04	0.04	0 12

any two adjoining planes at the beginning of the interval. Q is the heat transferred from plane to plane and δT the corresponding temperature rise. T is the final temperature at time t, and T_f is the temperature calculated from (7.14*d*).

11.19. The following comments may be made on the step calculations of Table 11.4:

1. It will be noted that the time intervals are taken progressively larger. This is a radical departure from the procedure of the Schmidt method. When the time interval is uniform and chosen according to the Schmidt scheme, the results of the two methods are identical.

2. By occasionally doubling the value of Δx as was done at $t = 38{,}000$ sec it is possible to speed the calculation greatly.

3. The results are in reasonably satisfactory agreement with those of classical theory. The step method gives results that are a bit too high for moderate distances from the surface and too low for greater distances. This is because the temperature gradient in the middle region is decreasing as time increases, so that the value used, which is that at the beginning of the interval, is larger than the average for the interval, which is the value that should really be used. At greater distances the reverse is true. A method of remedying this will be explained in the next problem. Thinner layers and shorter time intervals will of course give better results.

4. When the points show a tendency toward irregularity, it may be necessary to "smooth" the curve for any interval and then proceed from the smoothed curve. This is particularly necessary when the time intervals are chosen rather large.

5. It will be noted that the first half layer is in effect neglected. This is in keeping with the principle of the method that the temperature of each layer is that of its center, which, in the first half layer, is the surface. Only in very special cases, *e.g.*, some cases of spherical heat flow, does this introduce an error that need be considered.

6. When this is applied to a slab, it will be noted that the center plane gets heat from both sides; thus, its temperature rise is doubled. For this central plane we must accordingly use twice the temperature rise as calculated above—assuming,

TABLE 11.5.—STEP CALCULATIONS FOR THE COOLING OF A STEEL PLATE WHOSE THERMAL COEFFICIENTS DEPEND ON TEMPERATURE

A	B	C	D	E	F	G	H	I	J	K	L	M
Plane	Δt, hr	t at end of interval, hr	T at beginning of interval, °F	Estimated T at end of interval, °F	Estimated average T for interval, °F	ΔT between planes, °F	k, fph	Q $(= k\Delta t\Delta T/\Delta x)$, Btu/ft^2	$Q_m - Q_n$, Btu/ft^2	$c\rho$	δT $\left(= \frac{Q_m - Q_n}{c\rho\Delta x}\right)$, °F	T at end of interval, °F
0	0.01	0.01	1000	880	940	...	..	....	....	..	.....	880
1	"	"	1000	985	992	52	22	114	114	77	15	985
0	0.01	0.02	880	760	820	...	..	....	....	..	....	760
1	"	"	985	951	968	148	22	326	264	76	35	950
2	"	"	1000	992	996	28	22	62	62	77	8	992
0	0.01	0.03	760	640	700	...	..	....	....	..	....	640
1	"	"	950	900	925	225	23	518	384	75	51	899
2	"	"	992	980	986	61	22	134	108	76	14	978
3	"	"	1000	996	998	12	22	26	26	77	3	997
0	0.01	0.04	640	520	580	...	..	....	....	..	......	520
1	"	"	899	837	868	288	23	663	443	73	61	838
2	"	"	978	958	968	100	22	220	165	75	22	956
3	"	"	997	990	993	25	22	55	44	76	6	991
4	"	"	1000	996	998	5	22	11	11	77	2 × 1 5	997
0	0.01	0.05	520	400	460	...	..	....	....	..	. . .	400
1	"	"	838	766	802	342	24	822	502	71	71	767
2	"	"	956	926	941	139	23	320	221	74	30	926
3	"	"	991	981	986	45	22	99	82	76	11	980
4	"	"	997	991	994	8	22	17	17	77	2 × 2	993

0	0.02	0 07	400	280	340		. .	. . . .	. . . .	. .	. . . .	280
1	"	"	767	637	702	362	24	1736	881	70	126	641
2	"	"	926	850	888	186	23	855	525	73	72	854
3	"	"	980	946	963	75	22	330	242	75	32	948
4	"	"	993	973	983	20	22	88	88	77	2 × 11	971
0	0.03	0.10	280	100	190	. . .	. .	. . . .	. . . .	. .	. . . .	100
1	"	"	641	477	559	369	25	2766	1081	67	162	479
2	"	"	854	732	793	234	24	1685	884	72	123	731
3	"	"	948	870	909	116	23	801	589	74	80	868
4	"	"	971	911	941	32	22	212	212	75	2 × 28	915
0	0.03	0.13	100	0	30	. .	. .	. . . .	. . . .	. .	. . . .	0
1	"	"	479	331	405	375	25	2816	916	63	145	334
2	"	"	731	607	669	264	24	1900	835	68	123	608
3	"	"	868	766	817	148	24	1065	723	72	100	768
4	"	"	915	821	868	51	23	352	352	73	2 × 48	819
0	0.03	0.16	0	0	0	. . .	. .	. . . .	. . . .	. .	. . . .	0
1	"	"	334	260	297	297	26	2318	413	61	68	266
2	"	"	608	494	551	254	26	1905	753	66	114	494
3	"	"	768	654	711	160	24	1152	756	70	108	660
4	"	"	819	711	765	55	24	396	396	72	2 × 55	709
0	0.04	0.20	0	0	0	. . .	. .	. . . .	. . . .	. .	. . . .	0
1	"	"	266	210	238	238	26	2478	346	60	58	208
2	"	"	494	392	443	205	26	2132	662	64	103	391
3	"	"	660	519	590	147	25	1470	1000	67	149	511
4	"	"	709	569	639	49	24	470	470	68	2 × 69	571
0	0.05	0.25	0	0	0	. . .	. .	. . .	. . . .	. .	. . . .	0
1	"	"	208	152	180	180	26	2340	312	58	54	154
2	"	"	391	281	336	156	26	2028	665	62	107	284
3	"	"	511	379	445	109	25	1363	825	64	129	382
4	"	"	571	405	488	43	25	538	538	65	2 × 83	405

of course, symmetrical heating for the two faces. The next illustration furnishes an example of this.

7. A little study will show that in cases like this where the thermal constants are not dependent on temperature (over the range used), the process may be somewhat shortened by making more direct use of the diffusivity α.

11.20. Cooling of Armor Plate. We shall now apply the step method to a problem whose solution by other schemes—not involving electrical or other experimentation—would be of doubtful feasibility. A large plane steel plate 0.8 ft (9.6 in.) thick and at a uniform temperature of 1000°F has its surfaces cooled to 0°F at the rate of 200°F/min for the first 3 min and 100°F/min for the next 4 min. The thermal constants are assumed as follows: at 1000°F, $k = 22$, $c = 0.16$, $\rho = 480$ fph; at 0°F, $k = 27$, $c = 0.11$, $\rho = 490$, with an assumed linear variation between these temperatures. Temperatures inside the plate will be calculated for various times. The calculations would also hold without serious error for the range 1100 to 100°F.

We shall divide the plate by planes 0.1 ft apart, and, because of symmetry, it will be necessary to consider only half the thickness. To try to avoid the error, which would be rather serious in this case, mentioned in paragraph 3 of Sec. 11.19 we shall use the average temperature for any time interval. This involves no difficulty at the surface, but it is evident that for any other plane the final temperature calculated for any interval will be dependent on the average chosen. The best way to arrive at the estimated final and average temperature for any time interval is to plot the temperature curve for each interval as determined and then project it for the next. If the final values for the interval agree reasonably well with the projected or estimated values, the results may be considered satisfactory. The procedure involves trial and error and is in effect a relaxation method.

The step calculations for the first 15 min of cooling are given in Table 11.5 and some of the curves in Fig. 11.9. Column B of the table gives the time interval used, and C the total elapsed time at the end of each interval. ΔT in column G is the (aver-

age) temperature difference between planes, and Q in column I is the heat loss per square foot from the layer centering on any plane. J gives the net heat loss and L the temperature change. The doubled value for the center plane in column M has been

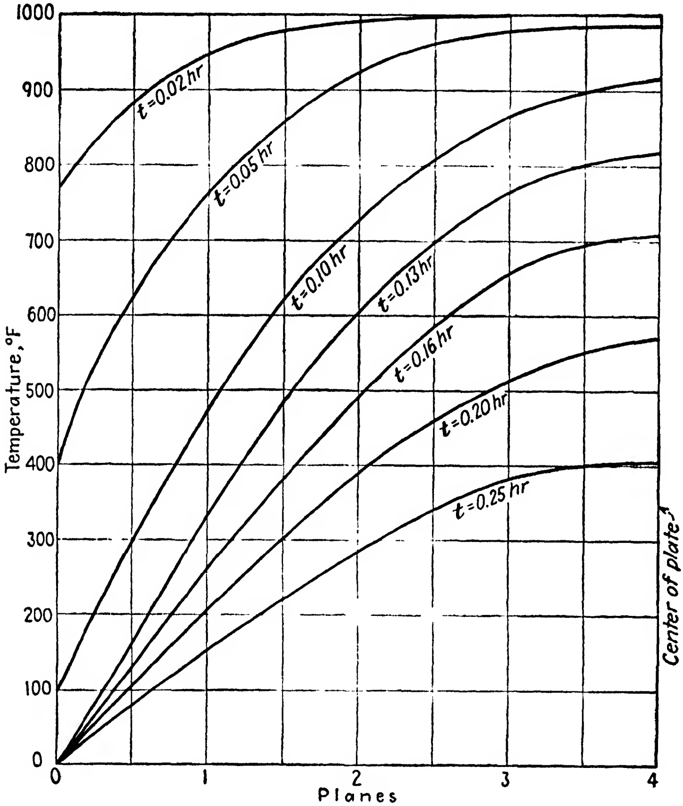

FIG. 11.9. Calculated cooling curves for a steel plate 0.8 ft thick with thermal coefficients dependent on temperature. See Sec. 11.20.

explained in comment 6, Sec. 11.19. When—and not until when—the values in column M agree closely with those in E, the results for any interval are considered satisfactory. In arriving at the estimated values for column E various expedients of the trained calculator may be found useful, such as making use of differences and, in particular, extrapolation of the curves of temperature vs. time for each of the planes. Smoothing—not necessary here—may be resorted to, to quicken the calculations.

It is evident from a glance at columns H and K that any calculation by classical methods involving the assumption of constant thermal coefficients would be considerably in error. It may also be remarked that cooling of the surfaces by radiation or by contact with a fluid, with known surface heat transfer coefficient, would not present any insuperable difficulty to the step method.

11.21. Heating of a Sphere. As a last illustration of the step method we shall calculate the temperatures in a sphere of

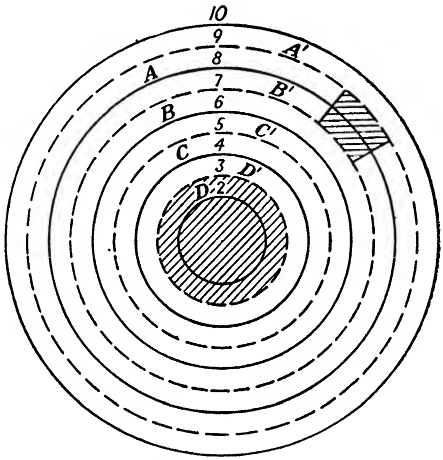

FIG. 11.10. Application of the step method to the problem of heating a sphere.

glass, initially at 0°C, whose surface is suddenly heated to 100°C. This is of interest as a case of three-dimensional flow whose results can be easily compared with classical theory.

Assume (see Fig. 11.10) $R = 10$ cm, $k = 0.0024$, $c = 0.161$, $\rho = 2.60$, $\alpha = 0.00573$ cgs. Imagine the sphere divided into layers 2 cm thick by spherical surfaces of radii 8, 6, 4, and 2 cm. We shall consider the heat flow from the surface to layer A ($r = 8$), then to B, and assume that the difference goes to warm a spherical shell 2 cm thick, centered (as regards thickness) on A. This shell would have radii 7 and 9 cm.

This case will be treated like the previous ones as essentially one of quasi-linear flow from layer to layer; we must accordingly find the mean area to use in calculating the heat flow from the surface to layer A, from this to B, etc. Consider the equation

for linear heat flow

$$\Delta Q = kA \frac{\Delta T}{\Delta x} \Delta t \tag{a}$$

and the equation [see (4.5k)] for heat flowing radially through a spherical shell

$$\Delta Q = \frac{4\pi k(T_1 - T_2)r_1 r_2 \Delta t}{r_1 - r_2} \tag{b}$$

If these two are equated, the average area A_m to be used in (a) is obtained. Considering that $r_1 - r_2$ is equivalent to Δx and $T_1 - T_2$ to ΔT, we have

$$A_m = 4\pi r_1 r_2 = \sqrt{4\pi r_1^2 \times 4\pi r_2^2} = \sqrt{A_1 A_2} \tag{c}$$

Using then the geometric means of the two areas, we have $A' = 4\pi \times 10 \times 8 = 4\pi \times 80$; $B' = 4\pi \times 48$; $C' = 4\pi \times 24$; and $D' = 4\pi \times 8$. Likewise, the volumes of the 2-cm thick spherical shells (shaded portion in Fig. 11.10) whose heating we have to consider are $V_A = \frac{4}{3}\pi(9^3 - 7^3) = \frac{4}{3}\pi \times 386$; $V_B = \frac{4}{3}\pi \times 218$; $V_C = \frac{4}{3}\pi \times 98$; $V_D = \frac{4}{3}\pi \times 27$. (Layer D is taken as the 3-cm radius core, which is assumed as uniform in temperature.) 4π may be canceled throughout and the areas taken as 80, 48, 24, and 8 cm², and the volumes as 128.7, 72.7, 32.7, and 9 cm³.

The step calculations are listed in Table 11.6, and the resultant temperature curves are given in Fig. 11.11. As in Table 11.5 the values in column E are the estimated temperatures for the end of each interval, giving therefore average values for the temperatures and temperature differences in F and G. Trial and error, with help from plotting, is used in arriving at values for column E such that they will be in fair agreement with the final temperatures as calculated in column M. Any estimated values for E that do not lead to such agreement must be discarded. If less pains are taken than in the present calculations and a larger departure between E and M allowed, fair results can still be obtained by smoothing the curves. It is to be noted that, as in the two previous illustrations, the first half layer is (effectively) neglected and is supposed to assume the surface temperature quickly.

The values in column N have been calculated from (9.16l)

TABLE 11.6.—STEP CALCULATIONS FOR THE HEATING OF A GLASS SPHERE, INITIALLY AT 0°C, WITH SURFACE AT 100°C. $R = 10$ CM, $k = 0.0024$, $c\rho = 0.418$ CGS

A	B	C	D	E	F	G	H	I	J	K	L	M	N
Layer	Δt, sec	t at end of interval, sec	T at beginning of interval, °C	Estimated T at end of interval, °C	Estimated average T, °C	ΔT between layers, °C	A_m, average area, cm²*	$Q\left(= \frac{kA_m\Delta t\Delta T}{\Delta r}\right)$, cal	$Q_m - Q_n$, cal	V, cm³*	$\delta T\left(= \frac{Q_m - Q_n}{c\rho V}\right)$, °C	T, °C	T_f (formula), °C
A	50	50	0	8.6	4.3	95.7	80	459	459	128.7	8.5	8.5	
A	50	100	8.5	16.0	12.3	87.7	80	421	387	128.7	7.1	15.7	
B	"	"	0	1.2	0.6	11.7	48	34	34	72.7	1.1	1.1	
A	100	200	15.7	27.7	21.7	78.3	80	752	643	128.7	11.9	27.6	
B	"	"	1.1	4.5	2.8	18.9	48	109	102	72.7	3.4	4.5	
C	"	"	0	0.6	0.3	2.5	24	7	7	32.7	0.5	0.5	
A	100	300	27.6	37.0	32.3	67.7	80	650	502	128.7	9.3	36.9	
B	"	"	4.5	8.5	6.5	25.8	48	148	132	72.7	4.3	8.8	
C	"	"	0.5	1.5	1.0	5.5	24	16	15	32.7	1.1	1.6	
D	"	"	0	0.2	0.1	0.9	8	1	1	9	0.3	0.3	
A	200	500	36.9	50.5	43.7	56.3	80	1082	733	128.7	13.7	50.6	49.5
B	"	"	8 8	18.0	13.4	30.3	48	349	291	72.7	9.6	18.4	16.0
C	"	"	1 6	5.2	3.4	10.0	24	58	53	32.7	3.9	5.5	3.0
D	"	"	0.3	1.5	0.9	2.5	8	5	5	9	1.3	1.6	0.5

TABLE 11.6.—(*Continued*)

A	B	C	D	E	F	G	H	I	J	K	L	M	N
Layer	Δt, sec	t at end of interval, sec	T at beginning of interval, °C	Estimated T at end of interval, °C	Estimated average T, °C	ΔT between layers, °C	A_m, average area, cm²*	Q $\left(=\dfrac{kA_m\Delta t\Delta T}{\Delta r}\right)$, cal	$Q_m - Q_n$, cal	V, cm³*	δT $\left(=\dfrac{Q_m - Q_n}{c\rho V}\right)$, °C	T, °C	T_f (formula), °C
A	400	900	50.6	66.6	58.6	41.4	80	1587	869	128.7	16.1	66.7	66.6
B	"	"	18.4	36.4	27.4	31.2	48	718	537	72.7	17.6	36.0	35.5
C	"	"	5.5	18.0	11.7	15.7	24	181	156	32.7	11.4	16.9	15.4
D	"	"	1.6	8.6	5.1	6.6	8	25	25	9	6.6	8.2	5.2
A	600	1,500	66.7	78.9	72.8	27.2	80	1564	634	128.7	11.8	78.5	78.7
B	"	"	36.0	55.8	45.9	26.9	48	930	597	72.7	19.7	55.7	55.8
C	"	"	16.9	36.3	26.6	19.3	24	333	274	32.7	20.0	36.9	36.8
D	"	"	8.2	24.4	16.3	10.3	8	59	59	9	15.7	23.9	23.3
A	1,000	2,500	78.5	88.1	83.3	16.7	80	1603	567	128.7	10.5	89.0	88.5
B	"	"	55.7	74.9	65.3	18.0	48	1036	587	72.7	19.3	75.0	75.5
C	"	"	36.9	62.5	49.7	15.6	24	449	341	32.7	25.0	61.9	63.5
D	"	"	23.9	52.9	38.4	11.3	8	108	108	9	28.7	52.6	53.7

* Note, however, the omission of the factor 4π.

and are seen to be in excellent agreement with the results of the step method. In this formula the values of r used are 8, 6, 4, and 1.5 for A, B, C, and D, respectively. From a practical standpoint one would, of course, hardly expect to use the step method on a problem involving constant thermal coefficients and conditions as simple as this. It turns out, however, that for the shorter times the step method may involve less labor

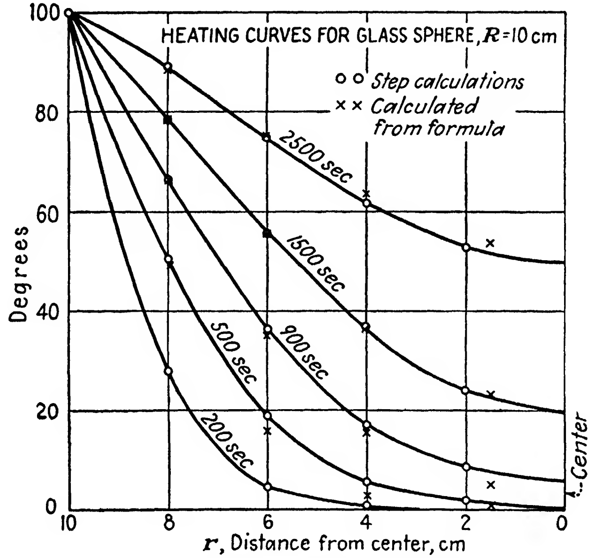

FIG. 11.11. Heating curves for a glass sphere of radius 10 cm, initially at 0°C, whose surface is quickly raised to 100°C. Calculations by the step method are seen to give results almost identical with those of formula (9.16l).

than the application of the formula, because of the number of terms required in the latter. In this case, smaller time intervals would have to be used. For the longer times the use of the formula is much simpler.

11.22. Cylindrical flow may be treated by the same principles as those used in the last case. The average areas to be used are logarithmic means defined by

$$A_m = \frac{A_1 - A_2}{2.303 \log_{10} (A_1/A_2)} \tag{a}$$

(If $A_1/A_2 < 1.4$, the above value is within 1 per cent of the arithmetic mean, which may accordingly be used.) The step

method has also been used with good success in treating a problem whose analytical solution* presents some difficulty. This is the case of the heat flow in an infinite solid bounded internally by a cylindrical surface of controlled temperature—a problem of practical interest in connection with the air conditioning of deep mines. In applying the step method to brick-shaped solids, rectangular bars, etc., the solution may be approximately obtained, as indicated in Sec. 11.11, by multiplying together solutions for the corresponding slab cases.

The step method should be particularly useful to geologists in making possible the treatment of all sorts of special problems such as the cooling of intrusions† of various sorts, either with or without generation of heat (as in the decomposition of granite). It would allow the treatment of cases where the temperature of the intrusion or the rate of heat generation is not uniform, or even where the intrusion and surrounding rocks are of different materials. While the step method is simplest to apply when the boundary temperatures are known, a little application of the trial-and-error principle should give an approximate solution of almost any problem of this sort, even if radiation cooling is involved.

* Smith.[135] See also Carslaw and Jaeger.[28] For graphs of the solution of this problem see Gemant.[44a]

† See Secs. 7.23, 8.9, and 9.3; also Lovering,[87] Boydell,[19] Berry,[13] and Van Orstrand.[152]

CHAPTER 12

METHODS OF MEASURING THERMAL-CONDUCTIVITY CONSTANTS

12.1. From the similarity between the flow of heat and of electricity it might be supposed that heat-conductivity measurements could be made with an accuracy approaching that of electrical conductivity. Unfortunately, this is by no means the case. Temperature difference and heat flow are not as easily and accurately measurable as their electrical analogues, potential difference and current. Furthermore, while we have almost perfect insulators for electricity, we do not have such for heat. The result is that thermal-conduction measurements are seldom of greater accuracy than one or two per cent probable error, and indeed the error is likely to be much larger than this unless great care is taken.

It is not proposed to give here an exhaustive account of methods of conductivity measurement but rather to limit the discussion to several standard methods and certain others that are interesting applications of the preceding theory. Those who wish to pursue the subject further may consult the articles dealing with heat-conductivity measurement in Glazebrook,[46] Winkelmann,[162] Kohlrausch,[78] or Roberts,[119] the surveys by Griffiths,[50] Ingersoll,[61] and Jakob,[67] and the modern discussions by Awbery,[3,4] Powell,[111,112,113,114], and Worthing and Halliday.[163]

12.2. The modern tendency in measuring thermal conductivity is toward greater directness than formerly. All that is necessary to determine this constant is a knowledge of the rate of heat flow through a given area of specimen under known temperature gradient. The heat is almost always produced electrically. The simplest and commonest arrangement involves flow in only one dimension. The chief difficulties here arise from heat losses, and these may be minimized by the use of silica

aerogel for insulation and by the employment of guard rings. (This means that the heat flow is measured only for a central portion of the area where it is uniform.) Radial-flow methods eliminate most of these losses but have difficulties of their own. Periodic or other variable-state (as regards temperature-time relations) methods have sometimes been used to give conductivity, but more generally diffusivity.

12.3. Linear Flow; Poor Conductors. The standard method here is to sandwich a flat electrically heated element between two similar flat slabs of the material under test, on the farther side of which are water-cooled plates. A guard ring is used to prevent losses that might otherwise be large. In one form of this apparatus (Griffiths[50,51]) usable for specimens up to almost a foot in thickness, the hot plate is 3 by 3 ft with a similarly heated guard ring 1 ft wide and separated from the central plate by a narrow air gap. The two cold plates and specimens are 5 by 5 ft, and surface temperatures are determined by thermocouples. The use cf the guard ring assures heat flow normal to the surface all over the central hot plate, 3 by 3 ft, whose energy input is measured. Apparatus of this general type is also used by our National Bureau of Standards.

In a small-scale apparatus of this type developed by Griffiths and Kaye[52] the specimens are 45 mm in diameter and 0.5 to 4 mm thick arranged on each side of an electrically heated copper disk, the outer surfaces being in contact with water-cooled copper blocks. Thermoelements give the temperature gradient. A guard ring is unnecessary. The method is well adapted to porous materials under definite pressure.

Birch and Clark[16] have measured the conductivity of various rocks by a variation of the preceding methods in which special care is taken to avoid certain errors. Instead of using two similar specimens, one on each side of the heating coil, only a single specimen is used at a time. To eliminate loss of heat the heater is surrounded by a "dome" that covers it and is kept at the same temperature as the heater. The rock specimen is 0.25 in. (6.35 mm) thick and 1.50 in. (38.1 mm) in diameter. It is surrounded by a guard ring of "isolantite" with outer diameter of 3 in. The cold plate, heater, and dome are all of

copper with heating coils in the last two. The temperature drop through the specimen is about 5°C, and the whole apparatus can be immersed in baths at temperatures up to 400°C or more. The special feature of the method is the use of atmospheres of nitrogen and helium that give thin films of these gases between the rock faces and copper plates; through these films the heat is conducted to or away from the rock faces. By measurement of the apparent conductivity in each gas it is possible to make the small correction for temperature discontinuity at the rock faces.

In a method useful for thin materials such as mica, the specimen is clamped between the ends of two copper bars, one of which carries a heating and the other a cooling coil. The heat flow is determined by measuring with thermocouples the temperature gradient along the bar, the conductivity of the copper bars being known. This method has also been developed so that it can be used at various points on a sheet of continuous material.

Comparison methods go back to Christiansen.[30] The specimen under test, which should be thin and a rather poor conductor, is placed between two plates of a material, *e.g.*, glass, whose conductivity is known. Thermocouples placed in thin copper sheets on each side of the glass plates, and thereby on each side of the specimen also, allow measurement of temperature gradients. If a steady heat flow is maintained normal to these surfaces, the conductivities of specimen and glass are inversely proportional to their temperature gradients. Sieg[131] and Van Dusen[151] have applied this method to small specimens, and the same principle has been made use of in the heat meter (Nicholls[104]). This is a thin plate of cork board or similar material of known conductivity with an array of thermocouples on each side, which can be applied to measure the heat loss from a wall.

12.4. Linear Flow; Bar Method—Metals. Of the many methods used to determine the thermal conductivity of metals one of the best[51] surrounds the bar with silica aerogel in a guard cylinder with heating and cooling coils on the ends. These maintain a temperature gradient in the cylinder the same as

that in the bar under test so that the radial and other losses are reduced to a minimum.

A very simple and usable, but only moderately accurate, method is that of Gray.[49] The specimen in the form of a bar 4 to 8 cm long and 2 to 4 mm in diameter has one end screwed into a copper block forming the bottom of a hot-water bath and the other into a 6-cm diameter copper sphere that serves as a calorimeter. Temperatures are determined by thermometers in the copper block and ball. Lateral losses are largely eliminated by a protective covering. For a description of the more complicated bar methods such as that of Jaeger and Diesselhorst[65] the reader is referred to the above mentioned surveys.

12.5. Radial Flow. When materials, particularly poor conductors, can be formed into cylinders or hollow spheres, the radial-flow method may be useful. This has the advantage of largely or even totally eliminating lateral heat losses, but the advantage gained may be lost through difficulties in temperature measurement. In the Niven[105] method the specimen is in the form of two half cylinders 9 cm in diameter and 15 cm or more long which are fitted together accurately. A known amount of heat per cm length is supplied by a resistance wire along the axis, and the temperatures at radial distances of, say, 1 and 3 cm are determined by thermocouples. From these data the conductivity is readily computed with the aid of the cylindrical flow equation. Hering[55] has suggested the use of hemispherical caps to avoid end losses in the cylindrical method.

A standard method of measuring the conductivity of some types of insulating material is to wrap the material about an electrically heated cylinder or pipe. The cylinder has an extension or guard ring at each end, and only the heat input to the central section is used in the measurement, thereby eliminating end losses.

In applying the spherical-flow method the material is formed into two closely fitting hemispherical shells of perhaps 8 cm internal diameter and 15 cm external, filled with oil or other liquid and immersed in a bath. In the cavity is a resistance coil that furnishes a known amount of heat and also a stirrer, whose energy input must also be taken into account. Thermo-

couples measure the two surface temperatures. In applying this method to iron, Laws, Bishop, and McJunkin[83] formed the thermoelements by electroplating the surfaces with copper and using copper leads.

The British Electrical and Allied Industries Research Association[20a] has developed a method for determining the thermal conductivity of soils based on (4.5p). Heat is electrically supplied at a measured rate to a buried copper sphere 3 to 9 in. in diameter, and the temperature of its surface measured after the steady state has been reached. This is useful for determining conductivity with a minimum of disturbance of the soil. It should also be easily possible to develop methods based on (9.5k), using a buried source, for the relatively quick "assaying" of soil in connection with heat-pump installations.

12.6. Diffusivity Measurements. Conductivity may be calculated from diffusivity measurements if specific heat and density are also determined. One method of measuring diffusivity is to have the material in the form of a plate or slab with a thermocouple buried in the center midway between the two faces. The slab is kept at constant temperature until the temperature is uniform throughout, and then the surfaces are suddenly chilled (or heated) by immersion in a stirred liquid bath, the center temperature changes being continuously recorded. With the aid of the equation for the unsteady-state linear flow in the slab the diffusivity is readily obtained. The method has also been applied[63] to measurements on sands or muds by packing them in a rectangular sheet-copper container with insulated edges. This is handled just as the slab above.

Diffusivity can also be measured by the periodic-flow method, by use of (5.3a). This involves a knowledge of the period and range of temperature at a given distance below the surface, the range at the surface being known. This last condition can be eliminated if the range is known for two or more distances. Forbes,[41]* who measured the annual variations of temperature for different depths of soil and rock near Edinburgh, was one of the first to determine thermal constants in this way.

* See also Kelvin,[146] "Mathematical and Physical Papers," III, p. 261.

12.7. Liquids and Gases. Some of the same methods applicable to measurements of conductivity in solids, *viz.*, heat transfer through a specimen from a hot to a cold plate, are also usable for liquids and gases. Convection* can be minimized by using small thicknesses and by having the heat flow downward. Absence of convection is shown if variations in thickness and temperature gradient have no effect on the final result. In gases convection may be considered to be eliminated if the apparent conductivity is independent of pressure.

Erk and Keller[38] in measuring the conductivity of glycerin-water mixtures used disks of fluid 11.7 cm in diameter and only 3 mm thick, but Bates,[10] by using special precautions, including the equivalent of a guard ring, was able to avoid convection even when the thickness was as great as 50 mm (diameter of central area, 12.7 cm). In measurements on gases the hot-wire method has certain advantages over the plate arrangement. The heat flows radially from a central hot wire to the surrounding cylinder. Sherratt and Griffiths,[126] in using this method on air, Freon, and other gases, have avoided some of the difficulties associated with it by using a thick platinum wire. This is arranged so that the energy input can be measured for the central section only, thus avoiding end effects.

* Radiation effects must be guarded against in heat-conduction measurements in general, even in the case of solids. See Johnston and Ruehr.[71]

APPENDIX A

VALUES OF THE THERMAL CONDUCTIVITY CONSTANTS

The following table has been compiled from a number of sources, including the "International Critical Tables,"[64] "Smithsonian Physical Tables,"[136] Landolt-Börnstein,[80] McAdams,[90] and others. Values are expressed in cgs or fph units (Sec. 1.5). Where temperature is not specified, ordinary room temperature may generally be assumed. Conversion factors are given in Table 1.2.

TABLE A.1.—THERMAL CONDUCTIVITY, SPECIFIC HEAT, DENSITY, AND THERMAL DIFFUSIVITY OF VARIOUS MATERIALS

Material	Average temp., °C	cgs				fph			
		$k \times 10^3$	c	ρ	$\alpha \times 10^3$	k	c	ρ	α
Metals:									
Aluminum	0	485	0.208	2.71	860	117	0.208	169	3.33
Antimony	0	44	0.049	6.65	135	10.6	0.049	415	0.52
Bismuth	0	20	0.029	9.80	70	4.9	0.029	612	0.28
Brass (yellow)	0	204				49			
Cadmium	18	222	0.055	8.65	467	54	0.055	540	1.82
Cobalt, 1.4% Fe, 1.1% Ni	30	165				40			
Copper	0	927	0.091	8.94	1140	224	0.091	558	4.42
Copper	400	868	0 096	8 76	1032	210	0.096	547	4.00
Everdur alloy*	20	79				19			
German silver	0	70				16.9			
Gold	18	700	0 030	19.3	1209	169	0.030	1204	4.68
Inconel†	70	36	0.109	8.55	38.6	8.7	0.109	534	0.15

* 96% Cu, 3% Si, 1% Mn (American Brass Co.).

† 79.5% Ni, 13% Cr, 6.5% Fe (International Nickel Co.).

TABLE A.1.—(*Continued*)

Material	Average temp., °C	cgs				fph			
		$k \times 10^3$	c	ρ	$\alpha \times 10^3$	k	c	ρ	α
Iron, pure	0	148	0.104	7.86	181	35.8	0.104	491	0.701
Iron, cast		112	0.11	7.6	134	27	0.11	474	0.52
Lead	18	83	0.030	11.3	245	20.1	0.030	705	0.95
Magnesium	50	376	0.232	1.74	932	91	0.232	109	3.60
Manganin	18	52				12.6			
Mercury	0	20.0	0.033	13.6	44.6	4.83	0.033	849	0.172
Molybdenum	0	346	0.059	10.2	575	83.7	0.059	636	2.23
Nickel	0	142	0.103	8.90	155	34.4	0.103	555	0.60
Nickel	200	132	0.120	8.84	125	32.0	0.120	552	0.48
Palladium	18	168	0.054	11.9	261	40.6	0.054	743	1.01
Platinum	18	166	0.032	21.46	242	40.2	0.032	1340	0.94
Silver	0	999	0.056	10.49	1700	242	0.056	655	6.60
Steel, mild	0	107	0.11	7.85	124	26	0.11	490	0.48
Steel, mild	600	87	0.16	7.68	71	21	0.16	479	0.27
Steel, 1% C	18	108	0.11	7.82	126	26	0.11	488	0.48
Steel, av. stainless		62				15			
Tin, white	0	157	0.054	7.30	398	38.0	0.054	456	1.55
Tungsten	0	380	0.032	19.3	615	92	0.032	1204	2.39
Tungsten	1600	250	0.039	18.9	339	60	0.039	1180	1.31
Tungsten	2800	310				75			
Zinc	0	268	0.091	7.14	413	65	0.091	446	1.60

Table A.1.—(*Continued*)

Material	Average temp., °C	cgs				fph			
		$k \times 10^3$	c	ρ	$\alpha \times 10^3$	k	c	ρ	α
Insulating materials:									
Aerogel, silica	0	0.050	0.20	0.14	1.79	0.012	0.20	8.5	0.007
Asbestos	0	0.36	0.25	0.58	2.48	0 087	0.25	36	0.010
Ashes, wood	50	0.17				0.041			
Cork, ground	40	0.10	0.48	0.13	1.6	0.024	0.48	8	0.006
Cotton, loose	30	0.10		0.08		0.024		5	
Cotton, fabric		0.19				0 046			
Charcoal, flakes	81	0.18		0.19		0.043		12	
Diatomaceous earth, powdered		0.11		0.18		0.027		11	
Glass wool, fine		0.08		0.02		0.02		1	
Loose fibrous materials, building ins.		0.08–0.12		0.03–0.16		0.02–0.03		2–10	
Mineral wool	30	0.09		0.16		0.022		10	
Magnesium carbonate (85%—pipe covering)		0.17				0.04			
Sawdust		0.14		0.19		0.03		12	
Wool, loose	0	0 09		0.08		0.022		5	
Rocks and building materials:									
Brick, fire clay	200	2 4	0.20	2.3	5.2	0.58	0.20	144	0.020
Brick masonry	20	1.5	0.20	1 7	4 4	0.36	0.20	106	0.017
Concrete, av. stone		2 2	0 20	2.3	4.8	0 54	0.20	144	0.019

TABLE A.1.—(*Continued*)

Material	Average temp., °C	cgs				fph			
		$k \times 10^3$	c	ρ	$\alpha \times 10^3$	k	c	ρ	α
Concrete, dams		5.8	0.22	2.47	10.7	1.4	0.22	154	0.041
Granite	0	6.5	0.19	2.7	12.7	1.6	0.19	168	0.050
Limestone	0	4.8	0.22	2.7	8.1	1.2	0.22	168	0.032
Marble		5.5	0.21	2.7	9.7	1.3	0.21	168	0.037
Sandstone		6.2	0.21	2.6	11.3	1.5	0.21	162	0.044
Traprock					7.5				0.029
Rock material av. for earth					10				0.04
Soils:									
Calcareous earth 43% water		1.7	0.53	1.67	1.9	0.41	0.53	104	0.007
Quartz sand, medium fine, dry		0.63	0.19	1.65	2.0	0.15	0.19	103	0.008
Quartz sand, 8.3% moisture		1.4	0.24	1.75	3.3	0.34	0.24	109	0.013
Sandy clay, 15% moisture		2.2	0.33	1.78	3.7	0.53	0.33	111	0.015
Soil, very dry		0.4–0.8			2–3	0.10–0.20			0.008–0.012
Some wet soils		3–8			4–10	0.8–2.0			0.02–0.04
Wet mud		2.0	0.60	1.50	2.2	0.50	0.60	94	0.008
Woods:									
Oak ⊥ grain	20	0.50		0.82		0.121		51	
Oak ‖ grain	20	0.86		0.82		0.208		51	
White pine ⊥ grain	20	0.23		0.50		0.056		31	
White pine ‖ grain	20	0.60		0.50		0.145		31	
Ash,[94] white ⊥ grain, 15.6% moisture	29	0.42	0.43	0.65	1.50	0.102	0.43	40	0.0057
Birch,[94] yellow ⊥ grain, 10.8% moisture	29	0.41	0.38	0.71	1.52	0.099	0.38	44	0.0059

Table A.1.—(*Continued*)

Material	Average temp., °C	cgs				fph			
		$k \times 10^3$	c	ρ	$\alpha \times 10^3$	k	c	ρ	α
Hemlock,[94] western, ⊥ grain, 23% moisture	29	0.33	0 49	0 54	1 25	0 080	0 49	34	0 0048
Oak,[94] red ⊥ grain, 12.4% moisture	29	0 46	0 41	0 70	1 60	0 111	0 41	44	0 0062
Pine,[94] so. yellow, ⊥ grain, 13.8% moisture	29	0.38	0.42	0 60	1 51	0 092	0 42	37	0 0059
Miscellaneous materials:									
Air, 1 atm	0	0.055	0.24	0 00129	178	0.0133	0.24	0.080	0.70
Beef, raw	50	1.2		–	1 3	0.29			0 005
Carbon, gas	20	8.5				2.1			
Fused quartz	−200	1.5	0.043	2.2	15 9	0 36	0 043	137	0 061
Fused quartz	0	3.4	0.166	2.2	9 3	0.82	0 166	137	0 036
Fused quartz	100	4.5	0.200	2.2	10 2	1.1	0 200	137	0 040
Glass, ordinary		2.1	0.16	2 6	5 1	0 51	0 16	162	0 020
Glass, pyrex		2.7	0.20	2 25	6 0	0 65	0 20	140	0 023
Hydrogen, 1 atm	0	0.38	3 41	0 00009	1240	0 092	3 41	0 0056	4 8
Ice	0	5.3	0 49	0.92	11 8	1 28	0 49	57	0.046
Rubber, hard (ebonite)	0	0.39		1 19		0 094		74	
Rubber, soft		0.40	0.45	1.1	0.8	0 10	0.45	69	0.003
Rubber, sponge		0 09				0 022			
Snow, fresh		0.3				0.07			
Water	0	1 32	1.01	1.00	1 31	0.32	1.01	62.4	0 0051
Water	80	1.64	1.00	0.97	1.69	0.40	1.00	60.5	0.0066
Water, an average commonly used		1.43	1.00	1.00	1.43	0.35	1.00	62.4	0.0056

TABLE A.2.—VALUES OF THE COEFFICIENT OF HEAT TRANSFER h*

	fph units, Btu/(hr)(ft²)(°F)	cgs units, cal/(sec)(cm²)(°C)
Air, heating or cooling	0.2–8	2.7×10^{-5} to 1.1×10^{-3}
Polished surface in still air, small temp. difference	1.3–1.7	1.8×10^{-4} to 2.3×10^{-4}
Blackened surface in still air, small temp. difference	1.8–2.5	2.5×10^{-4} to 3.3×10^{-4}
Surface in contact with oil, heating or cooling	10–300	1.4×10^{-3} to 4×10^{-2}
Surface in contact with water, heating or cooling	50–3000	7×10^{-3} to 0.4
Surface in contact with boiling water	300–9000	0.04–1.2

* From McAdams[90] and other sources.

APPENDIX B

INDEFINITE INTEGRALS

$$\int u\,dv = uv - \int v\,du$$

$$\int e^{ax}\,dx = \frac{e^{ax}}{a}$$

$$\int \frac{dx}{x} = \ln x$$

$$\int xe^{ax}\,dx = \frac{e^{ax}}{a^2}(ax - 1)$$

$$\int x^m\,dx = \frac{x^{m+1}}{m+1}, \text{ if } m \neq -1$$

$$\int a^{bx}\,dx = \frac{a^{bx}}{b \ln a}$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$\int (x^2 \pm a^2)^{1/2}\,dx = \frac{1}{2}[x\sqrt{x^2 \pm a^2} \pm a^2 \ln(x + \sqrt{x^2 \pm a^2})]$$

$$\int (a^2 - x^2)^{1/2}\,dx = \frac{1}{2}\left(x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\frac{x}{a}\right)$$

$$\int \sec^2 x\,dx = \tan x$$

$$\int x^2 \sin x\,dx = 2x \sin x - (x^2 - 2)\cos x$$

$$\int \tan x\,dx = -\ln \cos x$$

$$\int x^2 \cos x\,dx = 2x \cos x + (x^2 - 2)\sin x$$

$$\int x \sin ax\,dx = \frac{1}{a^2}(\sin ax - ax \cos ax)$$

$$\int \frac{dx}{x(a + bx)} = \frac{1}{a}\ln\frac{x}{a + bx}$$

$$\int x \cos ax\,dx = \frac{1}{a^2}(\cos ax + ax \sin ax)$$

$$\int \frac{dx}{\sqrt{a + bx}} = \frac{2\sqrt{a + bx}}{b}$$

$$\int \sin ax \sin bx\,dx = \frac{\sin(a - b)x}{2(a - b)} - \frac{\sin(a + b)x}{2(a + b)}, \; a \neq b$$

$$\int \sin ax \cos bx\,dx = -\frac{\cos(a - b)x}{2(a - b)} - \frac{\cos(a + b)x}{2(a + b)}, \; a \neq b$$

$$\int \cos ax \cos bx\,dx = \frac{\sin(a - b)x}{2(a - b)} + \frac{\sin(a + b)x}{2(a + b)}, \; a \neq b$$

$$\int \sin^2 ax\,dx = \frac{1}{2a}(ax - \sin ax \cos ax)$$

$$\int \cos^2 ax\,dx = \frac{1}{2a}(ax + \sin ax \cos ax)$$

$$\int e^{ax} \sin bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx)$$

$$\int e^{ax} \cos bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a \cos bx + b \sin bx)$$

$$\int \frac{e^{-x^2}}{x^2}\,dx = -\frac{e^{-x^2}}{x} - 2\int e^{-x^2}\,dx$$

$$\int x^2 e^{-x^2}\,dx = \frac{-xe^{-x^2}}{2} + \frac{1}{2}\int e^{-x^2}\,dx$$

$$\int x^m \ln x\,dx = x^{m+1}\left[\frac{\ln x}{m+1} - \frac{1}{(m+1)^2}\right]$$

APPENDIX C

DEFINITE INTEGRALS

$$\int_0^{\pi/2} \sin^n x \, dx = \int_0^{\pi/2} \cos^n x \, dx$$

$$\int_0^{\infty} \frac{\sin^2 x \, dx}{x^2} = \frac{\pi}{2}$$

$$\int_0^{\infty} \frac{\sin ax \, dx}{x} = \frac{\pi}{2}, \text{ if } a > 0; 0, \text{ if } a = 0; -\frac{\pi}{2}, \text{ if } a < 0$$

$$\int_0^{\infty} \frac{\sin x \cos ax \, dx}{x} = 0, \text{ if } a < -1 \text{ or } > 1; \frac{\pi}{4}, \text{ if } a = -1 \text{ or } +1;$$

$$\frac{\pi}{2}, \text{ if } 1 > a > -1$$

$$\int_0^{\infty} \cos (x^2) \, dx = \int_0^{\infty} \sin (x^2) \, dx = \frac{1}{2} \sqrt{\frac{\pi}{2}}$$

$$\int_0^{\pi} \sin ax \sin bx \, dx = \int_0^{\pi} \cos ax \cos bx \, dx = 0, \text{ if } a \neq b$$

$$\int_0^{\pi} \sin^2 ax \, dx = \int_0^{\pi} \cos^2 ax \, dx = \frac{\pi}{2}$$

$$\int_0^{\infty} e^{-a^2x^2} \, dx = \frac{1}{2a} \sqrt{\pi}$$

$$\int_0^{\infty} x^n e^{-ax} \, dx = \frac{n!}{a^{n+1}}$$

$$\int_0^{\infty} e^{-a^2x^2} \cos bx \, dx = \frac{\sqrt{\pi}}{2a} e^{-b^2/4a^2}, \text{ if } a > 0$$

$$\int_{-\infty}^{\infty} x^2 e^{-x^2} \, dx = 2 \int_0^{\infty} x^2 e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

APPENDIX D

TABLE D.1.—VALUES OF THE PROBABILITY INTEGRAL OR ERROR FUNCTION*

$$\Phi(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x e^{-\beta^2}\, d\beta = \frac{2}{\sqrt{\pi}} \int_{-x}^0 e^{-\beta^2}\, d\beta$$

x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$
0 00	0.00000	0.25	0.27633	0.50	0.52050
0 01	0 01128	0.26	0.28690	0.51	0.52924
0 02	0.02256	0.27	0.29742	0.52	0.53790
0 03	0.03384	0.28	0.30788	0.53	0.54646
0.04	0.04511	0.29	0.31828	0.54	0.55494
0.05	0.05637	0.30	0.32863	0.55	0.56332
0.06	0.06762	0.31	0.33891	0.56	0.57162
0.07	0.07886	0.32	0.34913	0.57	0.57982
0 08	0.09008	0.33	0.35928	0.58	0.58792
0.09	0.10128	0.34	0.36936	0.59	0.59594
0.10	0.11246	0.35	0.37938	0.60	0.60386
0.11	0.12362	0.36	0.38933	0.61	0.61168
0.12	0.13476	0.37	0.39921	0.62	0.61941
0.13	0.14587	0.38	0.40901	0.63	0.62705
0.14	0.15695	0.39	0.41874	0.64	0.63459
0.15	0.16800	0.40	0.42839	0.65	0.64203
0.16	0.17901	0.41	0.43797	0.66	0.64938
0.17	0.18999	0.42	0.44747	0.67	0.65663
0.18	0.20094	0.43	0.45689	0.68	0.66378
0.19	0.21184	0.44	0.46623	0.69	0.67084
0.20	0.22270	0.45	0.47548	0.70	0.67780
0.21	0.23352	0.46	0.48466	0.71	0.68467
0.22	0.24430	0.47	0.49375	0.72	0.69143
0.23	0.25502	0.48	0.50275	0.73	0.69810
0.24	0.26570	0.49	0.51167	0.74	0.70468

* From "Tables of Probability Functions," Vol. I, Bureau of Standards, Washington, 1941.[141]

TABLE D.1.—(*Continued*)

x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$
0.75	0.71116	1.10	0.88021	1.45	0.95970
0.76	0.71754	1.11	0.88353	1.46	0.96105
0.77	0.72382	1.12	0.88679	1.47	0.96237
0.78	0.73001	1.13	0.88997	1.48	0.96365
0.79	0.73610	1.14	0.89308	1.49	0.96490
0.80	0.74210	1.15	0.89612	1.50	0.96611
0.81	0.74800	1.16	0.89910	1.51	0.96728
0.82	0.75381	1.17	0.90200	1.52	0.96841
0.83	0.75952	1.18	0.90484	1.53	0.96952
0.84	0.76514	1.19	0.90761	1.54	0.97059
0.85	0.77067	1.20	0.91031	1.55	0.97162
0.86	0.77610	1.21	0.91296	1.56	0.97263
0.87	0.78144	1.22	0.91553	1.57	0.97360
0.88	0.78669	1.23	0.91805	1.58	0.97455
0.89	0.79184	1.24	0.92051	1.59	0.97546
0.90	0.79691	1.25	0.92290	1.60	0.97635
0.91	0.80188	1.26	0.92524	1.61	0.97721
0.92	0.80677	1.27	0.92751	1.62	0.97804
0.93	0.81156	1.28	0.92973	1.63	0.97884
0.94	0.81627	1.29	0.93190	1.64	0.97962
0.95	0.82089	1.30	0.93401	1.65	0.98038
0.96	0.82542	1.31	0.93606	1.66	0.98110
0.97	0.82987	1.32	0.93807	1.67	0.98181
0.98	0.83423	1.33	0.94002	1.68	0.98249
0.99	0.83851	1.34	0.94191	1.69	0.98315
1.00	0.84270	1.35	0.94376	1.70	0.98379
1.01	0.84681	1.36	0.94556	1.71	0.98441
1.02	0.85084	1.37	0.94731	1.72	0.98500
1.03	0.85478	1.38	0.94902	1.73	0.98558
1.04	0.85865	1.39	0.95067	1.74	0.98613
1.05	0.86244	1.40	0.95229	1.75	0.98667
1.06	0.86614	1.41	0.95385	1.76	0.98719
1.07	0.86977	1.42	0.95538	1.77	0.98769
1.08	0.87333	1.43	0.95686	1.78	0.98817
1.09	0.87680	1.44	0.95830	1.79	0.98864

Table D.1.—(*Continued*)

x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$
1.80	0.98909	2.10	0.99702 05	2 75	0.99989 94
1.81	0.98952	2.12	0.99728 36	2.80	0 99992 50
1.82	0 98994	2.14	0.99752 53	2 85	0.99994 43
1.83	0 99035	2.16	0.99774 72	2.90	0.99995 89
1.84	0.99074	2.18	0.99795 06	2.95	0.99996 98
1.85	0.99111	2.20	0.99813 72	3.00	0.99997 79095
1.86	0.99147	2.22	0.99830 79	3.10	0.99998 83513
1.87	0 99182	2.24	0 99846 42	3.20	0.99999 39742
1.88	0.99216	2.26	0.99860 71	3.30	0 99999 69423
1.89	0.99248	2.28	0.99873 77	3.40	0.99999 84780
1.90	0.99279	2.30	0.99885 68	3.50.	0.99999 92569
1.91	0.99309	2.32	0.99896 55	3 60	0.99999 96441
1.92	0 99338	2.34	0.99906 46	3.70	0.99999 98328
1.93	0.99366	2.36	0.99915 48	3.80	0 99999 99230
1.94	0.99392	2.38	0.99923 69	3.90	0.99999 99652
1.95	0 99418	2.40	0.99931 15	4.00	0.99999 99846
1.96	0.99443	2.42	0.99937 93	4.20	0.99999 99971
1.97	0 99466	2.44	0.99944 08	4.40	0.99999 99995
1.98	0.99489	2.46	0.99949 66	4.60	0.99999 99999
1.99	0 99511	2.48	0.99954 72	∞	1.00000
2.00	0.99532 23	2.50	0.99959 30		
2.02	0.99571 95	2.55	0.99968 93		
2.04	9 99608 58	2.60	0.99976 40		
2.06	0.99642 35	2.65	0.99982 15		
2.08	0 99673 44	2.70	0.99986 57		

APPENDIX E

TABLE E.1—VALUES OF e^{-x}*

These may be taken at once from an ordinary logarithm table as values of $1/10^{0.4343x}$, but the following abbreviated table may prove of occasional convenience:

x	e^{-x}	x	e^{-x}	x	e^{-x}
0.00†	1 000000	1.00	0.367879	4.00	0.018316
0.05	0.951229	1.10	0.332871	4.20	0.014996
0.10	0.904837	1 20	0 301194	4.40	0.012277
0.15	0.860708	1.30	0.272532	4.60	0.010052
0.20	0.818731	1.40	0.246597	4.80	0.008230
0.25	0.778801	1.50	0.223130	5.00	0.006738
0.30	0.740818	1.60	0.201897	5.50	0.004087
0.35	0.704688	1.70	0.182684	6.00	0 002479
0.40	0.670320	1.80	0.165299	6.50	0.001503
0.45	0.637628	1.90	0.149569	7.00	0.000912
0 50	0 606531	2 00	0.135335	7.50	0.000553
0 55	0.576950	2 20	0.110803	8.00	0.000335
0.60	0.548812	2.40	0.090718	8.50	0.000203
0.65	0.522046	2.60	0.074274	9.00	0.000123
0.70	0.496585	2.80	0.060810	9.50	0.000075
0.75	0.472367	3.00	0.049787	10.00	0.000045
0.80	0.449329	3.20	0.040762		
0.85	0.427415	3.40	0 033373		
0.90	0.406570	3.60	0.027324		
0.95	0.386741	3.80	0.022371		

* From "Smithsonian Physical Tables."[136]

† For very small x, $e^{-x} = 1 - x$.

APPENDIX F

TABLE F.1.—VALUES OF THE INTEGRAL

$$I(x) \equiv \int_x^\infty \beta^{-1} e^{-\beta^2}\, d\beta \left[= \frac{1}{2} \int_{x^2}^\infty \beta^{-1} e^{-\beta}\, d\beta = -\frac{1}{2} Ei(-x^2) \right]^*$$

x	$I(x)$	x	$I(x)$	x	$I(x)$
0.0001	8.9217	0.06	2.5266	0.31	0.9295
0.0002	8.2286	0.07	2.3731	0.32	0.9007
0.0003	7.8231	0.08	2.2403	0.33	0.8731
0.0004	7.5354	0.09	2.1234	0.34	0.8464
0.0005	7.3123	0.10	2.0190	0.35	0.8206
0.0006	7.1300	0.11	1.9247	0.36	0.7958
0.0007	6.9758	0.12	1.8388	0.37	0.7718
0.0008	6.8423	0.13	1.7600	0.38	0.7487
0.0009	6.7245	0.14	1.6873	0.39	0.7263
0.0010	6.6191	0.15	1.6197	0.40	0.7046
0.001	6.6191	0.16	1.5567	0.41	0.6836
0.002	5.9260	0.17	1.4977	0.42	0.6634
0.003	5.5205	0.18	1.4423	0.43	0.6437
0.004	5.2329	0.19	1.3900	0.44	0.6247
0.005	5.0097	0.20	1.3406	0.45	0.6062
0.006	4.8274	0.21	1.2938	0.46	0.5884
0.007	4.6733	0.22	1.2494	0.47	0.5711
0.008	4.5397	0.23	1.2072	0.48	0.5543
0.009	4.4220	0.24	1.1669	0.49	0.5380
0.010	4.3166	0.25	1.1285	0.50	0.5221
0.01	4.3166	0.26	1.0917	0.51	0.5068
0.02	3.6236	0.27	1.0565	0.52	0.4919
0.03	3.2184	0.28	1.0228	0.53	0.4774
0.04	2.9311	0.29	0.9904	0.54	0.4634
0.05	2.7084	0.30	0.9594	0.55	0.4498

* Computed from "Tables of Sine, Cosine and Exponential Integrals,"[142] Vols. I and II, and other sources. For $x < 0.2$, $I(x) = \ln \frac{1}{x} + \frac{x^2}{2} - \frac{x^4}{8} - 0.2886$.

TABLE F.1.—(*Continued*)

x	$I(x)$	x	$I(x)$	x	$I(x)$
0.56	0.4365	0.91	0.1476	1.65	0.009315
0.57	0.4237	0.92	0.1429	1.70	0.007508
0.58	0.4112	0.93	0.1383	1.75	0.006027
0.59	0.3990	0.94	0.1339	1.80	0.004818
0.60	0.3872	0.95	0.1295	1.85	0.003837
0.61	0.3758	0.96	0.1253	1.90	0.003042
0.62	0.3646	0.97	0.1212	1.95	0.002403
0.63	0.3538	0.98	0.1173	2.00	0.001890
0.64	0.3433	0.99	0.1134	2.05	0.001480
0.65	0.3331	1.00	0.1097	2.10	0.001154
0.66	0.3231	1.00	0.10969	2.15	8.963×10^{-4}
0.67	0.3134	1.02	0.10255	2.20	6.930 “
0.68	0.3041	1.04	0.09583	2.25	5.336 “
0.69	0.2949	1.06	0.08950	2.30	4.090 “
0.70	0.2860	1.08	0.08355	2.35	3.122 “
0.71	0.2774	1.10	0.07796	2.40	2.373 “
0.72	0.2690	1.12	0.07270	2.45	1.795 “
0.73	0.2609	1.14	0.06777	2.50	1.352 “
0.74	0.2529	1.16	0.06313	2.55	1.014 “
0.75	0.2452	1.18	0.05878	2.60	7.573×10^{-5}
0.76	0.2377	1.20	0.05470	2.65	5.629 “
0.77	0.2305	1.22	0.05088	2.70	4.166 “
0.78	0.2234	1.24	0.04730	2.75	3.069 “
0.79	0.2165	1.26	0.04394	2.80	2.251 “
0.80	0.2098	1.28	0.04081	2.85	1.643 “
0.81	0.2033	1.30	0.03787	2.90	1.194 “
0.82	0.1970	1.32	0.03512	2.95	8.641×10^{-6}
0.83	0.1909	1.34	0.03256	3.00	6.224 “
0.84	0.1849	1.36	0.03016	3.05	4.462 “
0.85	0.1791	1.38	0.02793	3.10	3.184 “
0.86	0.1735	1.40	0.02585		
0.87	0.1680	1.45	0.02123		
0.88	0.1627	1.50	0.01738		
0.89	0.1575	1.55	0.01417		
0.90	0.1525	1.60	0.01151		

APPENDIX G

TABLE G.1.—VALUES OF $S(x) = \frac{4}{\pi}\left(e^{-\pi^2 x} - \frac{1}{3}e^{-9\pi^2 x} + \frac{1}{5}e^{-25\pi^2 x} - \cdots\right)$

x	$S(x)$	x	$S(x)$	x	$S(x)$
0.001	1.0000	0.036	0.8752	0.071	0.6310
0.002	1.0000	0.037	0.8679	0.072	0.6249
0.003	1.0000	0.038	0.8605	0.073	0.6188
0.004	1.0000	0.039	0.8532	0.074	0.6128
0.005	1.0000	0.040	0.8458	0.075	0 6068
0.006	1.0000	0.041	0.8384	0.076	0.6009
0.007	1.0000	0.042	0.8310	0.077	0.5950
0.008	0.9998	0.043	0.8236	0.078	0.5892
0.009	0.9996	0.044	0.8162	0.079	0.5835
0.010	0.9992	0.045	0.8088	0.080	0.5778
0.011	0.9985	0.046	0.8015	0.081	0.5721
0.012	0.9975	0.047	0.7941	0.082	0.5665
0.013	0.9961	0.048	0.7868	0.083	0 5610
0 014	0.9944	0.049	0.7796	0.084	0.5555
0.015	0.9922	0.050	0.7723	0.085	0.5500
0.016	0.9896	0.051	0.7651	0.086	0.5447
0.017	0.9866	0.052	0.7579	0.087	0.5393
0.018	0.9832	0.053	0.7508	0.088	0.5340
0.019	0.9794	0.054	0.7437	0.089	0.5288
0 020	0.9752	0.055	0.7367	0.090	0.5236
0.021	0.9706	0.056	0.7297	0.091	0.5185
0 022	0.9657	0.057	0.7227	0.092	0.5134
0.023	0.9605	0.058	0.7158	0.093	0.5084
0.024	0.9550	0.059	0.7090	0.094	0.5034
0.025	0.9493	0.060	0.7022	0.095	0.4985
0.026	0.9433	0.061	0.6955	0.096	0.4936
0.027	0.9372	0.062	0.6888	0.097	0.4887
0.028	0.9308	0.063	0.6821	0.098	0.4839
0.029	0.9242	0.064	0.6756	0.099	0.4792
0.030	0.9175	0.065	0.6690	0.100	0.4745
0.031	0.9107	0.066	0.6626	0.102	0.4652
0.032	0.9038	0.067	0.6561	0.104	0.4561
0.033	0.8967	0.068	0.6498	0.106	0.4472
0.034	0.8896	0.069	0.6435	0.108	0.4385
0.035	0.8824	0.070	0.6372	0.110	0.4299

* From Olson and Schultz[108] and other sources.

TABLE G.1.—(*Continued*)

x	$S(x)$	x	$S(x)$	x	$S(x)$
0.112	0.4215	0.182	0.2113	0.36	0.0365
0.114	0.4133	0.184	0.2071	0.37	0.0330
0.116	0.4052	0.186	0.2031	0.38	0.0299
0.118	0.3973	0.188	0.1991	0.39	0.0271
0.120	0.3895	0.190	0.1952	0.40	0.0246
0.122	0.3819	0.192	0.1914	0.42	0.0202
0.124	0.3745	0.194	0.1877	0.44	0.0166
0.126	0.3671	0.196	0.1840	0.46	0.0136
0.128	0.3600	0.198	0.1804	0.48	0.0112
0.130	0.3529	0.200	0.1769	0.50	0.0092
0.132	0.3460	0.205	0.1684	0.52	0.0075
0.134	0.3393	0.210	0.1602	0.54	0.0062
0.136	0.3326	0.215	0.1525	0.56	0.0051
0.138	0.3261	0 220	0.1452	0 58	0.0042
0.140	0.3198	0.225	0.1382	0.60	0.0034
0.142	0.3135	0.230	0.1315	0.62	0.0028
0.144	0.3074	0.235	0.1252	0.64	0.0023
0.146	0.3014	0.240	0.1192	0.66	0.0019
0.148	0.2955	0.245	0.1134	0.68	0.0016
0.150	0.2897	0.250	0.1080	0.70	0.0013
0.152	0.2840	0.255	0.1028	0.72	0.0010
0.154	0.2785	0.260	0.0978	0.74	0.0009
0.156	0.2731	0.265	0.0931	0.76	0.0007
0.158	0.2677	0.270	0.0886	0.78	0 0006
0.160	0.2625	0.275	0.0844	0.80	0.0005
0.162	0.2574	0.280	0.0803	0.82	0.0004
0.164	0.2523	0.285	0.0764	0.84	0.0003
0.166	0.2474	0.290	0.0728	0.86	0.0003
0.168	0.2426	0.295	0.0693	0.88	0.0002
0.170	0.2378	0.300	0.0659	0.90	0.0002
0.172	0.2332	0.31	0.0597	0.92	0.0001
0.174	0.2286	0.32	0.0541	0.94	0.0001
0.176	0.2241	0.33	0.0490	0.96	0.0001
0.178	0.2198	0.34	0.0444	0.98	0.0001
0.180	0.2155	0.35	0.0402	1.00	0.0001

APPENDIX H

TABLE H.1.—VALUES OF $B(x) \equiv 2(e^{-x} - e^{-4x} + e^{-9x} - \cdots)$ AND $B_a(x) \equiv \frac{6}{\pi^2}\left(e^{-x} + \frac{1}{4}e^{-4x} + \frac{1}{9}e^{-9x} + \cdots\right)$

x	$B(x)$	$B_a(x)$	x	$B(x)$	$B_a(x)$	x	$B(x)$	$B_a(x)$
0.00	1.0000	1.0000	0.70	0.8752	0.3113	2.00	0.2700	0.0823
0.02	1.0000	0.8537	0.72	0.8643	0.3045	2.10	0.2445	0.0745
0.04	1.0000	0.7967	0.74	0.8531	0.2980	2.20	0.2213	0.0674
0.06	1.0000	0.7543	0.76	0.8418	0.2916	2.30	0.2003	0.0610
0.08	1.0000	0.7195	0.78	0.8303	0.2854	2.40	0.1813	0.0552
0.10	1.0000	0.6897	0.80	0.8186	0.2794	2.50	0.1641	0.0499
0.12	1.0000	0.6632	0.82	0.8068	0.2735	2.60	0.1485	0.0452
0.14	1.0000	0.6394	0.84	0.7950	0.2678	2.70	0.1344	0.0409
0.16	1.0000	0.6176	0.86	0.7831	0.2622	2.80	0.1216	0.0370
0.18	1.0000	0.5976	0.88	0.7711	0.2567	2.90	0.1100	0.0335
0.20	1.0000	0.5789	0.90	0.7591	0.2513	3.00	0.0996	0.0303
0.22	0.9999	0.5615	0.92	0.7471	0.2461	3.20	0.0815	0.0248
0.24	0.9998	0.5451	0.94	0.7351	0.2410	3.40	0.0667	0.0203
0.26	0.9995	0.5296	0.96	0.7232	0.2360	3.60	0.0546	0.0166
0.28	0.9990	0.5149	0.98	0.7112	0.2312	3.80	0.0447	0.0136
0.30	0.9983	0.5010	1.00	0.6994	0.2264	4.00	0.0366	0.0111
0.32	0.9972	0.4877	1.05	0.6700	0.2150	4.50	0.0222	0.0068
0.34	0.9957	0.4750	1.10	0.6413	0.2042	5.00	0.0135	0.0041
0.36	0.9938	0.4629	1.15	0.6132	0.1940	5.50	0.0082	0.0025
0.38	0.9913	0.4513	1.20	0.5860	0.1844	6.00	0.0050	0.0015
0.40	0.9883	0.4401	1.25	0.5596	0.1752	6.50	0.0030	0.0009
0.42	0.9846	0.4294	1.30	0.5340	0.1665	7.00	0.0018	0.0006
0.44	0.9804	0.4190	1.35	0.5095	0.1583	7.50	0.0011	0.0003
0.46	0.9755	0.4090	1.40	0.4858	0.1505	8.00	0.0007	0.0002
0.48	0.9700	0.3994	1.45	0.4631	0.1431	8.50	0.0004	0.0001
0.50	0.9639	0.3901	1.50	0.4413	0.1360			
0.52	0.9573	0.3810	1.55	0.4204	0.1293			
0.54	0.9500	0.3723	1.60	0.4005	0.1230			
0.56	0.9422	0.3639	1.65	0.3814	0.1170			
0.58	0.9339	0.3557	1.70	0.3631	0.1112			
0.60	0.9251	0.3477	1.75	0.3457	0.1058			
0.62	0.9158	0.3400	1.80	0.3291	0.1006			
0.64	0.9062	0.3325	1.85	0.3133	0.0957			
0.66	0.8962	0.3252	1.90	0.2981	0.0910			
0.68	0.8858	0.3181	1.95	0.2837	0.0866			

APPENDIX I

TABLE I.1.—BESSEL FUNCTIONS

x	$J_0(x)$	$J_1(x)$	x	$J_0(x)$	$J_1(x)$	x	$J_0(x)$	$J_1(x)$
0.0	1.00000	0.00000	4.0	−0.39715	−0.06604	8.0	0.17165	0.23464
0.1	0.99750	0.04994	4.1	−0.38867	−0.10327	8.1	0.14752	0.24761
0.2	0.99002	0.09950	4.2	−0.37656	−0.13865	8.2	0.12222	0.25800
0.3	0.97763	0.14832	4.3	−0.36101	−0.17190	8.3	0.09601	0.26574
0.4	0.96040	0.19603	4.4	−0.34226	−0.20278	8.4	0.06916	0.27079
0.5	0.93847	0.24227	4.5	−0.32054	−0.23106	8.5	0.04194	0.27312
0.6	0.91200	0.28670	4.6	−0.29614	−0.25655	8.6	0.01462	0.27275
0.7	0.88120	0.32900	4.7	−0.26933	−0.27908	8.7	−0.01252	0.26972
0.8	0.84629	0.36884	4.8	−0.24043	−0.29850	8.8	−0.03923	0.26407
0.9	0.80752	0.40595	4.9	−0.20974	−0.31469	8.9	−0.06525	0.25590
1.0	0.76520	0.44005	5.0	−0.17760	−0.32758	9.0	−0.09033	0.24531
1.1	0.71962	0.47090	5.1	−0.14433	−0.33710	9.1	−0.11424	0.23243
1.2	0.67113	0.49829	5.2	−0.11029	−0.34322	9.2	−0.13675	0.21741
1.3	0.62009	0.52202	5.3	−0.07580	−0.34596	9.3	−0.15766	0.20041
1.4	0.56686	0.54195	5.4	−0.04121	−0.34534	9.4	−0.17677	0.18163
1.5	0.51183	0.55794	5.5	−0.00684	−0.34144	9.5	−0.19393	0.16126
1.6	0.45540	0.56990	5.6	0.02697	−0.33433	9.6	−0.20898	0.13952
1.7	0.39798	0.57777	5.7	0.05992	−0.32415	9.7	−0.22180	0.11664
1.8	0.33999	0.58152	5.8	0.09170	−0.31103	9.8	−0.23228	0.09284
1.9	0.28182	0.58116	5.9	0.12203	−0.29514	9.9	−0.24034	0.06837
2.0	0.22389	0.57672	6.0	0.15065	−0.27668	10.0	−0.24594	0.04347
2.1	0.16661	0.56829	6.1	0.17729	−0.25586	10.1	−0.24903	0.01840
2.2	0.11036	0.55596	6.2	0.20175	−0.23292	10.2	−0.24962	−0.00662
2.3	0.05554	0.53987	6.3	0.22381	−0.20809	10.3	−0.24772	−0.03132
2.4	0.00251	0.52019	6.4	0.24331	−0.18164	10.4	−0.24337	−0.05547
2.5	−0.04838	0.49709	6.5	0.26009	−0.15384	10.5	−0.23665	−0.07885
2.6	−0.09680	0.47082	6.6	0.27404	−0.12498	10.6	−0.22764	−0.10123
2.7	−0.14245	0.44160	6.7	0.28506	−0.09534	10.7	−0.21644	−0.12240
2.8	−0.18504	0.40971	6.8	0.29310	−0.06522	10.8	−0.20320	−0.14217
2.9	−0.22431	0.37543	6.9	0.29810	−0.03490	10.9	−0.18806	−0.16035
3.0	−0.26005	0.33906	7.0	0.30008	−0.00468	11.0	−0.17119	−0.17679
3.1	−0.29206	0.30092	7.1	0.29905	0.02515	11.1	−0.15277	−0.19133
3.2	−0.32019	0.26134	7.2	0.29507	0.05433	11.2	−0.13299	−0.20385
3.3	−0.34430	0.22066	7.3	0.28822	0.08257	11.3	−0.11207	−0.21426
3.4	−0.36430	0.17923	7.4	0.27860	0.10963	11.4	−0.09021	−0.22245
3.5	−0.38013	0.13738	7.5	0.26634	0.13525	11.5	−0.06765	−0.22838
3.6	−0.39177	0.09547	7.6	0.25160	0.15921	11.6	−0.04462	−0.23200
3.7	−0.39923	0.05383	7.7	0.23456	0.18131	11.7	−0.02133	−0.23330
3.8	−0.40256	0.01282	7.8	0.21541	0.20136	11.8	0.00197	−0.23228
3.9	−0.40183	−0.02724	7.9	0.19436	0.21918	11.9	0.02505	−0.22898

TABLE I.2.—ROOTS OF $J_n(x) = 0$

Root number	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
1	2.40483	3.83171	5.13562	6.38016	7.58834	8.77148
2	5.52008	7.01559	8.41724	9.76102	11.06471	12.33860
3	8.65373	10.17347	11.61984	13.01520	14.37254	15.70017
4	11.79153	13.32369	14.79595	16.22346	17.61597	18.98013
5	14.93092	16.47063	17.95982	19.40941	20.82693	22.21780
6	18.07106	19.61586	21.11700	22.58273	24.01902	25.43034
7	21.21164	22.76008	24.27011	25.74817	27.19909	28.62662
8	24.35247	25.90367	27.42057	28.90835	30.37101	31.81172
9	27.49348	29.04683	30.56920	32.06485	33.53714	34.98878
10	30.63461	32.18968	33 71652	35.21867	36.69900	38.15987

APPENDIX J

TABLE J.1.—VALUES OF $C(x)^* = 2\left[\frac{e^{-xz_1^2}}{z_1J_1(z_1)} + \frac{e^{-xz_2^2}}{z_2J_1(z_2)} + \cdots\right]$, WHERE $z_1, z_2, \ldots$ ARE ROOTS OF $J_0(z_m) = 0$

x	$C(x)$	x	$C(x)$	x	$C(x)$
0.005	1.0000	0.205	0.4875	0.41	0.1496
0.010	1.0000	0.210	0.4738	0.42	0.1412
0.015	1.0000	0.215	0.4605	0.43	0.1332
0.020	1.0000	0.220	0.4475	0.44	0.1258
0.025	0.9999	0.225	0.4349	0.45	0.1187
0.030	0.9995	0.230	0.4227	0.46	0.1120
0.035	0.9985	0.235	0.4107	0.47	0.1057
0.040	0.9963	0.240	0.3991	0.48	0.0998
0.045	0.9926	0.245	0.3878	0.49	0.0942
0.050	0.9871	0.250	0.3768	0.50	0.0887
0.055	0.9798	0.255	0.3662	0.52	0.0792
0.060	0.9705	0.260	0.3558	0.54	0.0704
0.065	0.9596	0.265	0.3457	0.56	0.0628
0.070	0.9470	0.270	0.3359	0.58	0.0560
0.075	0.9330	0.275	0.3263	0.60	0.0499
0.080	0.9177	0.280	0.3170	0.62	0.0444
0.085	0.9015	0.285	0.3080	0.64	0.0396
0.090	0.8844	0.290	0.2993	0.66	0.0352
0.095	0.8666	0.295	0.2908	0.68	0.0314
0.100	0.8484	0.300	0.2825	0.70	0.0280
0.105	0.8297	0.305	0.2744	0.72	0.0249
0.110	0.8109	0.310	0.2666	0.74	0.0222
0.115	0.7919	0.315	0.2590	0.76	0.0198
0.120	0.7729	0.320	0.2517	0.78	0.0176
0.125	0.7540	0.325	0.2445	0.80	0.0157
0.130	0.7351	0.330	0.2375	0.85	0.0117
0.135	0.7164	0.335	0.2308	0.90	0.0088
0.140	0.6980	0.340	0.2242	0.95	0.0066
0.145	0.6798	0.345	0.2178	1.00	0.0049
0.150	0.6618	0.350	0.2116	1.05	0.0037
0.155	0.6442	0.355	0.2056	1.10	0.0028
0.160	0.6269	0.360	0.1997	1.15	0.0021
0.165	0.6100	0.365	0.1940	1.20	0.0016
0.170	0.5934	0.370	0.1885	1.25	0.0012
0.175	0.5771	0.375	0.1831	1.30	0.0009
0.180	0.5613	0.380	0.1779	1.35	0.0007
0.185	0.5458	0.385	0.1728	1.40	0.0005
0.190	0.5306	0.390	0.1679	1.50	0.0003
0.195	0.5159	0.395	0.1631	1.60	0.0002
0.200	0.5015	0.400	0.1585	1.70	0.0001

* Mainly from Olson and Schultz.[106]

APPENDIX K

MISCELLANEOUS FORMULAS

$e = 2.71828$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad (x^2 < \infty)$$

$$\ln(1 + x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \cdots \qquad (x^2 < 1)$$

$$\log_e x = \log_a x \cdot \log_e a = 2.3026 \log_{10} x$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad (x^2 < \infty)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad (x^2 < \infty)$$

$$e^{ix} = \cos x + i \sin x$$

$$\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

$$\sinh x = \frac{1}{2}(e^x - e^{-x})$$

$$\frac{d}{db}\int_a^b f(x)\,dx = f(b)$$

$$\cosh x = \frac{1}{2}(e^x + e^{-x})$$

$$\frac{d}{da}\int_a^b f(x)\,dx = -f(a)$$

$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$\frac{d}{dc}\int_a^b f(x,c)\,dx = \int_a^b \frac{\partial}{\partial c} f(x,c)\,dx + f(b,c)\frac{db}{dc} - f(a,c)\frac{da}{dc}$$

$$\int_a^b f(x)\,dx = (b - a)f(\beta), \text{ where } a < \beta < b$$

$$f(x + h) = f(x) + hf'(x + \beta h), \text{ where } 0 < \beta < 1$$

$$\sin x \sin y = \tfrac{1}{2}\cos(x - y) - \tfrac{1}{2}\cos(x + y)$$

$$\cos x \cos y = \tfrac{1}{2}\cos(x - y) + \tfrac{1}{2}\cos(x + y)$$

$$\sin x \cos y = \tfrac{1}{2}\sin(x - y) + \tfrac{1}{2}\sin(x + y)$$

APPENDIX L

THE USE OF CONJUGATE FUNCTIONS FOR ISOTHERMS AND LINES OF HEAT FLOW IN TWO DIMENSIONS*

In the elementary theory of complex analytic functions it is easily proved that if $f(z) = u(x,y) + iv(x,y)$ is an analytic function of the complex variable $z = x + iy$, and thus has a definite derivative with respect to z, then u and v, which are the real and imaginary parts of $f(z)$, must be related by the Cauchy-Riemann differential equations

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned} \tag{a}$$

Because of this interrelation, u and v are called *conjugate functions.* The pair have the following interesting properties which can be derived immediately from (*a*):

1. Both u and v satisfy the same differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0; \qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \tag{b}$$

2. The equations $u(x,y) = c_1$ and $v(x,y) = c_2$ represent two families of curves in the xy plane which are orthogonal to each other. For at any point (x,y) (where the denominators are not zero)

$$\left(\frac{dy}{dx}\right)_{u=c_1} = -\frac{\dfrac{\partial u}{\partial x}}{\dfrac{\partial u}{\partial y}} = \frac{\dfrac{\partial v}{\partial y}}{\dfrac{\partial v}{\partial x}} = \frac{-1}{\left(\dfrac{dy}{dx}\right)_{v=c_2}} \tag{c}$$

which is the well-known condition for such curves to cross each other orthogonally. That is, the slope of one curve is the negative reciprocal of the slope of the other.

3. When either u or v is known, its conjugate function can be obtained by integration. If u is known, we integrate the exact differential expression

$$dv \left(\equiv \frac{\partial v}{\partial x}\,dx + \frac{\partial v}{\partial y}\,dy\right) = -\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy \tag{d}$$

and if v is known we integrate

$$du \left(\equiv \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy\right) = \frac{\partial v}{\partial y}\,dx - \frac{\partial v}{\partial x}\,dy \tag{e}$$

* See, *e.g.*, Jeans,[69a, p. 261] Carslaw and Jaeger,[27a, p. 343] and Livens.[86a, p. 104]

The conditions that must be fulfilled to make both of the above exact differentials are satisfied by (a). Obviously, if the same function is used in one case for u and in another case for v, the derived conjugate functions in the respective cases will differ only in sign (neglecting any constant).

The above properties of conjugate functions have been utilized for the solution of two-dimensional problems in other fields than heat conduction, in particular that of electrical potential.

Let us now derive the conjugate function U for the heat-conduction problem of Sec. 4.4. Put in (e) $u = U$ and $v = T$, which is known, *viz.*,

$$T = \frac{2}{\pi} \tan^{-1} \left(\frac{\sin x}{\sinh y} \right) \tag{f}$$

Then we have

$$dU = \frac{2}{\pi} \left[\left(\frac{-\sin x/\cosh y}{1 - (\cos x/\cosh y)^2} \right) dx + \left(\frac{-\cos x \sinh y/\cosh^2 y}{1 - (\cos x/\cosh y)^2} \right) dy \right] \tag{g}$$

$$= \frac{2}{\pi} \left[\frac{\partial}{\partial x} \tanh^{-1} \left(\frac{\cos x}{\cosh y} \right) dx + \frac{\partial}{\partial y} \tanh^{-1} \left(\frac{\cos x}{\cosh y} \right) dy \right] \tag{h}$$

which is readily verified. Hence our solution for the conjugate function to T is

$$U = \frac{2}{\pi} \tanh^{-1} \left(\frac{\cos x}{\cosh y} \right) \tag{i}$$

It may be added that since (i) satisfies $(4.1a)$, this function might be taken to represent temperature, and its conjugate function (f) would then give the lines of heat flow. But the resulting temperature boundary conditions would differ accordingly and would represent quite a different physical situation from the problem treated in Sec. 4.1 Another application of the above results is found in Problem 6, Sec. 9.46.

APPENDIX M

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